Interaction graphs of Hamiltonian automata networks

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Introduction

Automata networks

For all $q \geq 2$, $[\![q]\!] = \{0, 1, \ldots, q-1\}$ is an **alphabet** of size q.

Definition (Automata network)

An **automata network** is a function $f : [\![q]\!]^n \to [\![q]\!]^n$ for some $q \geq 2$.

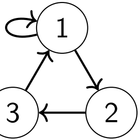
x	f(x)
000	000
001	100
010	001
011	101
100	010
101	010
110	011
111	011

Global function

$$f(x) = (f_1(x), f_2(x), f_3(x))$$

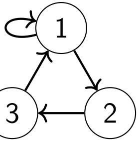
$$f_1(x) = \neg x_1 \wedge x_3$$
$$f_2(x) = x_1$$
$$f_3(x) = x_2$$

Local functions



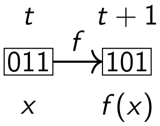
Interaction graph

Automata networks

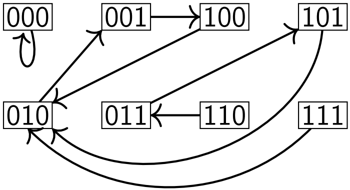


$f_1(x) = \neg x_1 \wedge x_3$
$f_2(x) = x_1$
$f_3(x) = x_2$

Sychronous update:



Transition graph:



Hamiltonian automata networks

Definition (period)

Let $f : [\![q]\!]^n \to [\![q]\!]^n$ bijective. The **period** ρ of f is the smallest word $(f^{\,0}(0^n), f^{\,1}(0^n), \ldots, f^{\,\ell}(0^n))$ such that $f^{\,\ell+1}(0^n) = 0^n$.

The period length $|\rho| = \ell + 1$ of a bijective function is between 1 and q^n.

Definition (Hamiltonian automata networks)

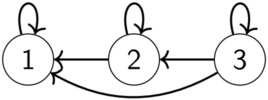
The automata network $f : [\![q]\!]^n \to [\![q]\!]^n$ is **Hamiltonian** if $f^{\,q^n}(0^n) = 0^n$ and $f^{\,t}(0^n) \neq 0^n$ for all $0 < t < q^n$.

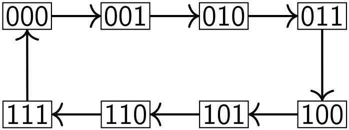
Equivalently:

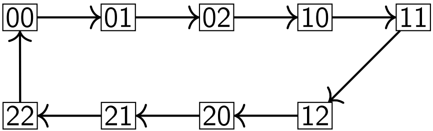
- the transition graph of f is a single limit cycle of length q^n;
- $|\rho| = q^n$;
- In ρ, there is exactly one time each configuration of $[\![q]\!]^n$.

Example: counter

$$f_i(x) = \begin{cases} x_i + 1 & \text{if } x_{[i+1,n]} = (q-1)^{n-i} \\ x_i & \text{otherwise.} \end{cases}$$







Remark

- The interaction digraph has about $n^2/2$ arcs.
- It has a maximum in-degree of n.

Example: de brujin sequence

Definition (de brujin sequence)

A de Bruijn sequence w of order n on a size-q alphabet is a cyclic sequence in which every possible length-n string occurs exactly once as a substring.

In other word, for all $x \in [\![q]\!]^n$, there exists $t \in [\![q^n]\!]$, such that $w_{[t,t+n \mod q^n]} = x$.

Theorem (Tatyana van Aardenne-Ehrenfest, 1951)

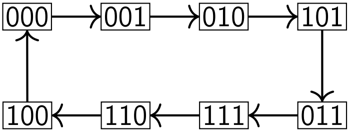
There exists $\frac{(q!)^{q^{n-1}}}{q^n}$ de brujin sequence of order n on a size-q alphabet.

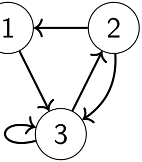
Example: de brujin sequence

Example of brujin sequence ($n = 3, q = 2$):

$$00010111.$$

We can define f such that $f(w_{[t,t+n]}) = w_{[t+1,t+n+1]}$ (addition done modulo q^n).





Remark

In this example,

- The interaction digraph has $2n - 1$ arcs.
- It has a maximum in-degree of n.

Questions

In this presentation, we are interested in these questions:

Question

What can be said about the interaction graph of a Hamiltonian automata network?

Question

What can be said about the interaction graph of a Hamiltonian automata network with a given alphabet size?

In the 'ingeneria' thesis of Arturo Zapata, the following question was asked:

Question

In the Boolean case, is the maximum in-degree of the interaction graph of a Hamiltonian automata network always n? (when $n \geq 3$)

Basic properties

Property 1

A digraph if **coverable** if there exists a set of vertex disjoint cycles that can cover all the vertices of the graph. Equivalently, a graph is coverable if $|N_G^+(S)| \geq |S|$ for all $S \subseteq V(G)$.

Property 1:

The interaction graph of a Hamiltonian automata network are coverable.

Remark

A Hamiltonian automata network $f : [\![q]\!]^n \to [\![q]\!]^n$ is bijective.

Theorem (Maximilien Gadouleau, 2018)

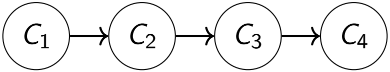
If an automata network is bijective, then its interaction graph is coverable.

Property 2:

A graph G is *unilaterally connected* if for every vertex u and v of G, either there is a path from u to v or from v to u in G.

Property 2:

The interaction graph of a Hamiltonian automata network is unilaterally connected.



Sub network

If there are no arcs from $V(G) \setminus I$ to I in G then $f_{|I} : [\![q]\!]^{|I|} \to [\![q]\!]^{|I|}$ is the restriction of f: for all $x \in [\![q]\!]^n$, $f_I(x) = f_{|I}(x_I)$.

Proposition

Consider a Hamiltonian $f \in F(G)$ and $I \subseteq V(G)$ such that there are no arcs from $V \setminus I$ to I in G. Then, $f_{|I}$ is Hamiltonian.

Proof:

- $f_{|I}$ is bijective. Indeed, if $f_{|I}$ is not bijective:
  - there exists $y \in [\![q]\!]^{|I|}$ without pre-images by $f_{|I}$.
  - Let $x \in [\![q]\!]^n$ with $x_I = y$ and $x_{V(G)\setminus I} = 0^{|V(G)\setminus I|}$.
  - x has no pre-images in f.
  - So f is not bijective which is absurd.

Sub network

If there are no arcs from $V(G) \setminus I$ to I in G then $f_{|I} : [\![q]\!]^{|I|} \to [\![q]\!]^{|I|}$ is the restriction of f: for all $x \in [\![q]\!]^n$, $f_I(x) = f_{|I}(x_I)$.

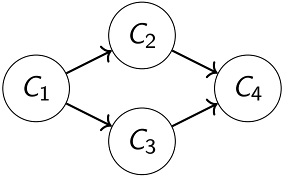
Proposition

Consider a Hamiltonian $f \in F(G)$ and $I \subseteq V(G)$ such that there are no arcs from $V \setminus I$ to I in G. Then, $f_{|I}$ is Hamiltonian.

Proof:

- $f_{|I}$ is bijective.
- $f_{|I}$ is Hamiltonian. Indeed, if $f_{|I}$ is not Hamiltonian:
    - there exists $y \in [\![q]\!]^{|I|}$ not in the period of $f_{|I}$.
    - Let $x \in [\![q]\!]^n$ with $x_I = y$ and $x_{V(G)\setminus I} = 0^{|V(G)\setminus I|}$.
    - x is not in the period of f.
    - So f is not Hamiltonian which is absurd.

Property 2



Proof:

- $f_{|C_1 \cup C_2}$ and $f_{|C_1 \cup C_3}$ are Hamiltonian so their period are $q^{|C_1 \cup C_2|}$ and $q^{|C_1 \cup C_3|}$.
- So, $f_{|C_1 \cup C_2 \cup C_3}$ has period $\mathrm{lcm}(q^{|C_1 \cup C_2|}, q^{|C_1 \cup C_3|}) < q^{|C_1 \cup C_2 \cup C_3|}$
- So, $f_{|C_1 \cup C_2 \cup C_3}$ is not Hamiltonian which is absurd.

Property 3

Definition (index of cyclicity)

The **index of cyclicity** $c(G)$ of G is the $\gcd$ of all the length of its cycles.

Property 3:

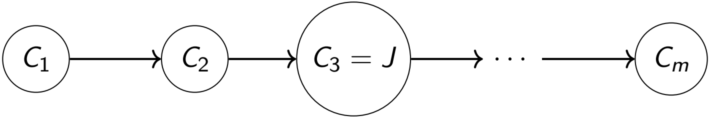
Let $f \in F(G, q)$ be Hamiltonian. Let J be a component of G
Then, $c(G[J]) = 1$ except if $q = |J| = 2$.

A (directed) graph is **cyclically** k**-partite** if its vertex set can be partitioned into k parts $J_0, \dots, J_{k-1}$ such that every arc of G goes from J_i to $J_{i+1 \mod k}$ for some $0 \leq i \leq k-1$.

Theorem (Brualdi et al., 1991)

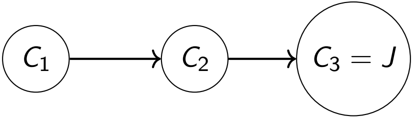
If $c(G) = k$, then G is cyclically k-partite.

Property 3: proof



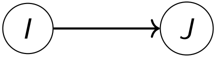
For all $1 \leq i \leq m$, $f_{|C_1 \cup \cdots \cup C_i}$ is Hamiltonian. So suppose $C_m = J$.

Property 3: proof



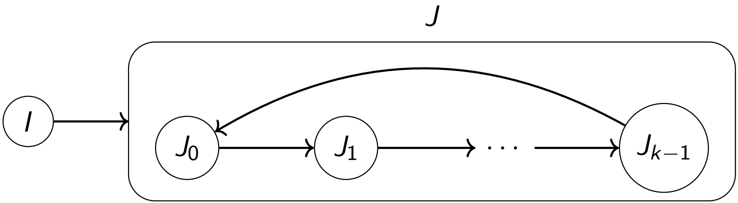
Let $I = C_1 \cup \cdots \cup C_{m-1}$.

Property 3: proof



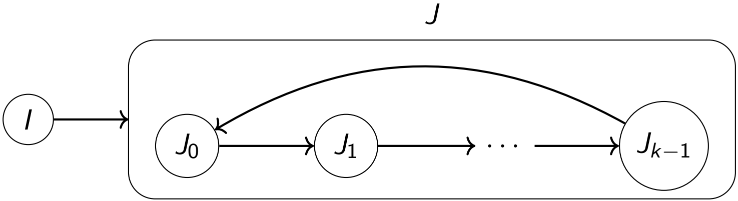
Let $k = c(G[J\,])$. So J is k-partite.

Property 3: proof



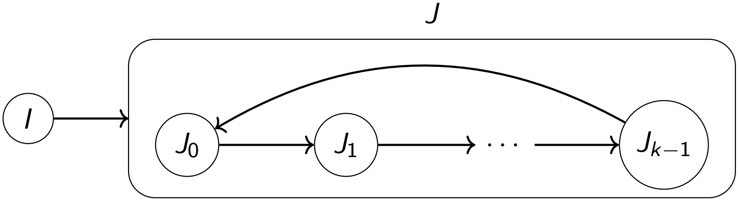
	I	J_0	J_1	J_2	...
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	

Property 3: proof



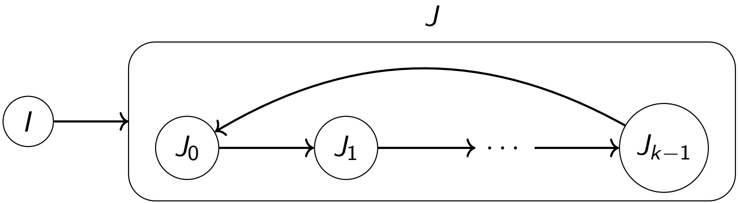
	I	J_0	J_1	J_2	...
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f(x)$	y^1				
$f^2(x)$	y^2				

Property 3: proof



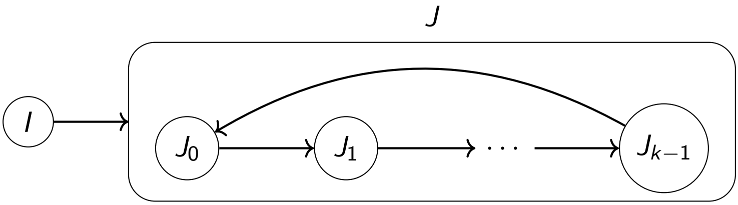
	I	J_0	J_1	J_2	...
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f(x)$	y^1				
$f^2(x)$	y^2				
$f^{q^{\lvert I\rvert}-1}(x)$	$y^{q^{\lvert I\rvert}-1}$				
$f^{q^{\lvert I\rvert}}(x)$	y^0				

Property 3: proof



	I	J_0	J_1	J_2	...
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	

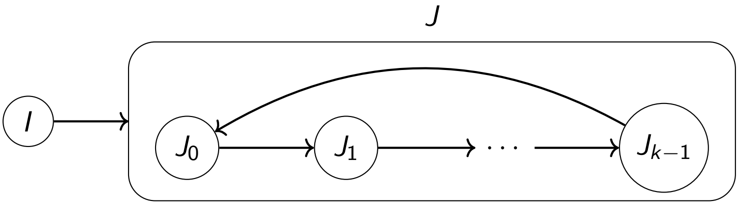
Property 3: proof



	I	J_0	J_1	J_2	$\cdots$
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f(x)$	y^1		$h_{J_1}^{(y^0)}(z_{J_0})$		

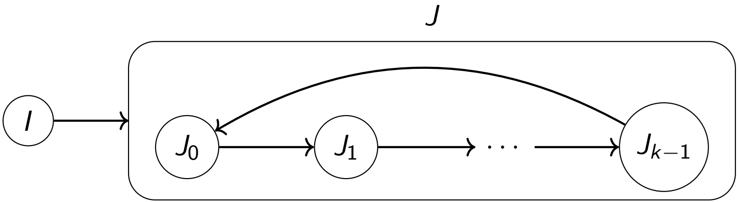
- Note that $N_G^+(J_{i-1}) \subseteq J_i$ and $N_G^-(J_i) \subseteq I \cup J_i$.
- $h_{J_i}^{(y)}$ is injective.
- Therefore, $|J_0| \leq |J_1| \leq \cdots \leq |J_{k-1}| \leq |J_0|$.
- Thus, $|J_0| = |J_1| = \cdots = |J_{k-1}| = |J|/k$.
- Therefore, $h_{J_i}^{(y)}$ is a permutation!

Property 3: proof



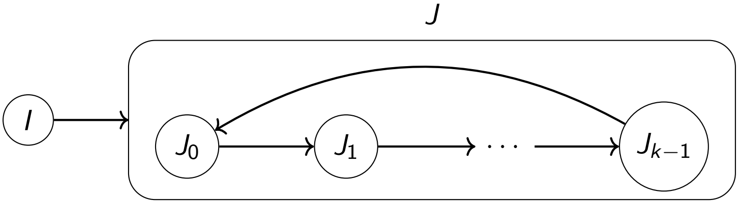
	I	J_0	J_1	J_2	$\cdots$
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f(x)$	y^1		$h_{J_1}^{(y^0)}(z_{J_0})$		
$f^2(x)$	y^2			$h_{J_2}^{(y^1)} \circ h_{J_1}^{(y^0)}(z_{J_0})$	

Property 3: proof



	I	J_0	J_1	J_2	$\cdots$
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f(x)$	y^1		$h_{J_1}^{(y^0)}(z_{J_0})$		
$f^2(x)$	y^2			$h_{J_2}^{(y^1)} \circ h_{J_1}^{(y^0)}(z_{J_0})$	
$f^k(x)$	y^k	$h_{J_0}^{(y^{k-1})} \circ \cdots \circ h_{J_1}^{(y^0)}(z_{J_0})$			

Property 3: proof



	I	J_0	J_1	J_2	$\cdots$
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f(x)$	y^1		$h_{J_1}^{(y^0)}(z_{J_0})$		
$f^2(x)$	y^2			$h_{J_2}^{(y^1)} \circ h_{J_1}^{(y^0)}(z_{J_0})$	
$f^k(x)$	y^k	$h_{J_0}^{(y^{k-1})} \circ \cdots \circ h_{J_1}^{(y^0)}(z_{J_0})$			
$f^{kq^{\lvert I \rvert}}(x)$	y^0	$g_0(z_{J_0})$	$g_1(z_{J_1})$	$g_2(z_{J_2})$	

$$g_i = \bigcirc_{t=0}^{kq^{|I|}-1} h_{J_{t \bmod k}}^{(y^{t \bmod q^{|I|}})}.$$

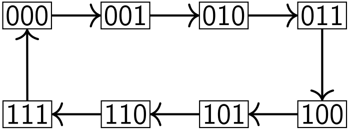
g_i is a permutation!

Property 3: proof

	I	J_0	J_1	J_2	$\dots$
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f^{kq^{\lvert I\rvert}}(x)$	y^0	$g_0(z_{J_0})$	$g_1(z_{J_1})$	$g_2(z_{J_2})$	

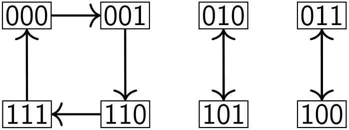
p_i : length of the smallest limit cycle of g_i.

g_0 $(p_0 = 8)$:



$z_{J_0} = 000$

g_1 $(p_1 = 2)$:



$z_{J_0} = 010$

$$p_i = q^{|J_0|} \text{ or } p_i \leq q^{|J_0|}/2.$$

Property 3: proof

	I	J_0	J_1	J_2	$\dots$
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f^{kq^{\lvert I \rvert}}(x)$	y^0	$g_0(z_{J_0})$	$g_1(z_{J_1})$	$g_2(z_{J_2})$	

Let $p = \mathrm{lcm}(p_0, \dots, p_{k-1})$. We know that $f^{pkq^{|I|}}(x) = x$. Therefore,

$$q^n \leq pkq^{|I|}$$
$$q^{|I|+k|J_0|} \leq pkq^{|I|}$$
$$q^{k|J_0|} \leq pk$$

Property 3: proof

	I	J_0	J_1	J_2	$\dots$
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f^{kq^{\|I\|}}(x)$	y^0	$g_0(z_{J_0})$	$g_1(z_{J_1})$	$g_2(z_{J_2})$	

Let $p = \mathrm{lcm}(p_0, \dots, p_{k-1})$.

$$q^{k|J_0|} \leq pk$$

If all $p_i \leq q^{|J_0|}/2$,

$$q^{k|J_0|} \leq k(q^{|J_0|}/2)^k$$
$$q^{k|J_0|} \leq kq^{k|J_0|}/2^k$$
$$1 \leq k/2^k$$

No solutions!

Property 3: proof

	I	J_0	J_1	J_2	$\dots$
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f^{kq^{\lvert I\rvert}}(x)$	y^0	$g_0(z_{J_0})$	$g_1(z_{J_1})$	$g_2(z_{J_2})$	

Let $p = \mathrm{lcm}(p_0, \dots, p_{k-1})$.

$$q^{k|J_0|} \leq pk$$

Suppose $p_0 = q^{|J_0|}$. If $p_i = q^{|J_0|}$ then $\mathrm{lcm}(p_0, p_i) = q^{|J_0|}$. So,

$$p \leq q^{|J_0|}(q^{|J_0|}/2)^k$$

Thus,

$$q^{k|J_0|} \leq kq^{k|J_0|}/2^{k-1}$$
$$1 \leq k/2^{k-1}$$

Solutions: $k \in \{1, 2\}$!

Property 3: proof

	I	J_0	J_1	J_2	$\dots$
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f^{kq^{\lvert I \rvert}}(x)$	y^0	$g_0(z_{J_0})$	$g_1(z_{J_1})$	$g_2(z_{J_2})$	

$$q^{k|J_0|} \leq pk$$

Suppose $p_0 = q^{|J_0|}$ and $k = 2$. $p = \mathrm{lcm}(p_0, p_1)$. If $\mathrm{lcm}(p_0, p_1) \neq q^{|J_0|}$. Then $p_1 \leq q^{|J_0|}/2 - 1$. So,

$$q^{2|J_0|} \leq 2q^{|J_0|}(q^{|J_0|}/2 - 1)$$
$$q^{|J_0|} \leq q^{|J_0|} - 2$$
$$0 \leq -2$$

No solution!

Property 3: proof

	I	J_0	J_1	J_2	$\dots$
x	y^0	z_{J_0}	z_{J_1}	z_{J_2}	
$f^{kq^{\|I\|}}(x)$	y^0	$g_0(z_{J_0})$	$g_1(z_{J_1})$	$g_2(z_{J_2})$	

$$q^{k|J_0|} \leq pk$$

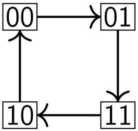
If $\mathrm{lcm}(p_0, p_1) = q^{|J_0|}$. Then,

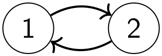
$$q^{2|J_0|} \leq 2q^{|J_0|}$$
$$q^{|J_0|} \leq 2$$

Unique solution: $|J_0| = 1$ and $q = 2$.

Exception: negative C_2

$f \in \{0,1\}^2 \to \{0,1\}^2$ with $f(x_1, x_2) = (x_2, x_1 + 1 \mod 2)$.





Properties

Properties

Let $f : [\![q]\!]^n \to [\![q]\!]^n$ be a Hamiltonian automata network and G be its interaction graph. Then,

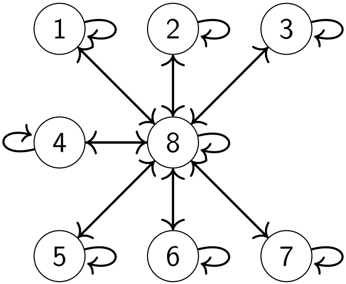
- G is coverable;
- G is unilaterally connected;
- for each component C of G, $c(G[C]) = 1$ except if $q = |C| = 2$.

Conjecture

If G has the three properties, then there exist a Hamiltonian automata network $f \in F(G, q)$ for some $q \geq 2$.

Properties

Star graph S_7:



Remark

S_n respect all the properties. However, if $q^{(q^2)} < n$ then the interaction digraph of a Hamiltonian function $f : [\![q]\!]^n \to [\![q]\!]^n$ is not S_n.

Reminder about permutations

Reminder about permutations

- A permutation is a bijective function $p : S \to S$.
- $(a \leftrightarrow b)$ is the *transposition* defined by

$$(a \leftrightarrow b) : x \mapsto \begin{cases} b & \text{if } x = a; \\ a & \text{if } x = b; \\ x & \text{otherwise.} \end{cases}$$

- $(a \leftrightarrow a)$ is the identity function.
- For all distinct $a_0, \dots, a_{m-1}$, $(a_0 \to a_1 \to \dots a_{m-1} \to)$ is the *cyclic permutation* defined by

$$(a_0 \to a_1 \to \dots a_{m-1} \to)(x) = \begin{cases} a_{(i+1) \mod m} & \text{if } x = a_i; \\ x & \text{otherwise.} \end{cases}$$

Reminder about permutations

- It is know that

$$(a_0 \to a_1 \to \dots a_{m-1} \to) = (a_0 \leftrightarrow a_{m-1}) \circ \cdots \circ (a_0 \leftrightarrow a_2) \circ (a_0 \leftrightarrow a_1).$$

  So any cyclic permutation on m elements can be expressed as the composition of $m-1$ transpositions.
- The number of transpositions is even iff m is odd!
- Any permutation can be expressed as the composition of cyclic permutations and so as the composition of transpositions.
- A permutation is even (*resp.* odd) if it is the composition of an even (*resp.* odd) number of transpositions (not counting the identity transpositions).
- It is known that a permutation cannot be even and odd at the same time.
- In fact, a permutation is even iff in its cycle decomposition, there is an even number of cyclic permutations of even size.

Odd alphabets

Odd alphabets

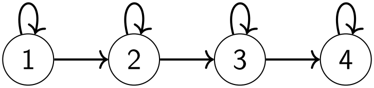
Theorem

The automata network $f : [\![q]\!]^n \to [\![q]\!]^n$ defined by:

$$\forall i \in [n], f_i(x) = \begin{cases} x_i + 1 \mod q & \text{if } i = 1; \\ (0 \leftrightarrow x_{i-1})(x_i) & \text{otherwise.} \end{cases}$$

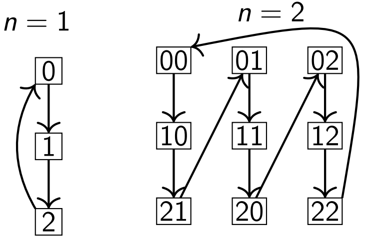
is Hamiltonian if q is odd.

The interaction graph G has maximum in-degree 2 for all n!



Odd alphabet

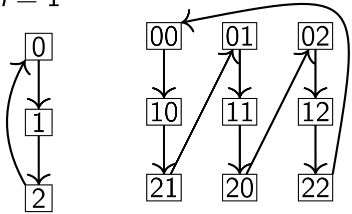
$q = 3$:



Odd alphabet

$q = 3$:

$n = 1$ $\qquad$ $n = 2$



t	y^t
0	0
1	1
2	2

t	x^t	y^t	y^{t+3}	y^{t+6}
0	0	0	1	2
1	1	0	1	2
2	2	1	0	2

Odd alphabets

$n = 2$

t	x^t	y^t	y^{t+3}	y^{t+6}
0	0	0	1	2
1	1	0	1	2
2	2	1	0	2

$n = 3$

t	x^t	y^t	y^{t+9}	y^{t+18}
0	00	0	1	2
1	10	0	1	2
2	21	0	1	2
3	01	1	0	2
4	11	0	1	2
5	20	1	0	2
6	02	1	0	2
7	12	1	2	0
8	22	1	0	2

odd alphabets

$n = 1$

t	y^t
0	0
1	1
2	2
3	3
4	4

$n = 2$

t	x^t	y^t	y^{t+5}	y^{t+10}	y^{t+15}	y^{t+20}
0	0	0	1	2	3	4
1	1	0	1	2	3	4
2	2	1	0	2	3	4
3	3	1	2	0	3	4
4	4	1	2	3	0	4

$n = 3$

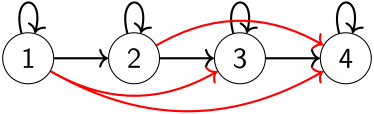
t	x^t	y^t	y^{t+25}	y^{t+50}	y^{t+75}	y^{t+100}
0	00	0	1	2	3	4
1	10	0	1	2	3	4
2	21	0	1	2	3	4
3	31	1	0	2	3	4
4	41	0	1	2	3	4
5	01	1	0	2	3	4
6	11	0	1	2	3	4
7	20	1	0	2	3	4
8	32	1	0	2	3	4
9	42	1	2	0	3	4
10	02	1	0	2	3	4
11	12	1	2	0	3	4
12	22	1	0	2	3	4
13	30	1	2	0	3	4
14	43	1	2	0	3	4
15	03	1	2	3	0	4
16	13	1	2	0	3	4
17	23	1	2	3	0	4
18	33	1	2	0	3	4
19	40	1	2	3	0	4
20	04	1	2	3	0	4
21	14	1	2	3	4	0
22	24	1	2	3	0	4
23	34	1	2	3	4	0
24	44	1	2	3	0	4

Even alphabets

Even alphabets

Theorem

Let $f \in F(G, q)$ be a Hamiltonian automata network with q even and $n \geq 3$. Let $C_1, \ldots, C_m$ be the topologically ordered components of G. Then, for all $1 \leq i \leq m$, $N^-_G(C_i) = C_1 \cup \cdots \cup C_i$.



Theorem

Let $f \in F(G, q)$ be a Hamiltonian automata network with q even and $n \geq 3$. Let $C_1, \ldots, C_m$ be the topologically ordered components of G. If $\{v\}$ is a feedback vertex set of C_i for some $1 \leq i \leq m$ then $N^-_G(v) = C_1 \cup \cdots \cup C_i$.

Even alphabets

Lemma

Let $f \in F(G, q)$ be a permutation with q even. Let $I \subseteq V(G)$. If $|I| = |N_G^+(I)|$ and $N_G^-(N_G^+(I)) \neq V(G)$. Then f has an even number of cycles.

- Even permutation $\Leftrightarrow$ Even number of even cycles.
- But the sum of the cycles is q^n which is even.
- Therefore, there is an even number of odd cycles.
- So, even number of cycles $\Leftrightarrow$ even number of even cycles $\Leftrightarrow$ even permutation.

Even alphabets

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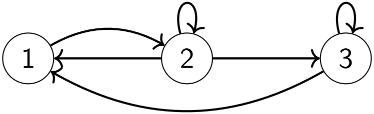

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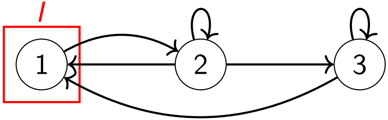

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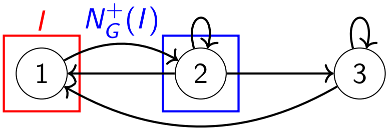

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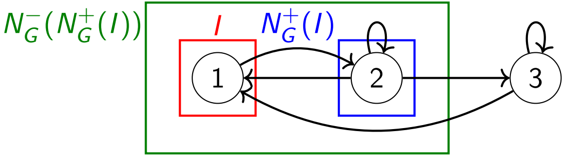
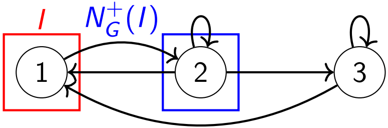
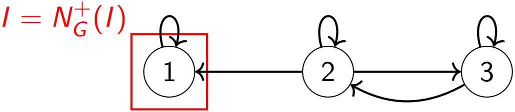


Even alphabets

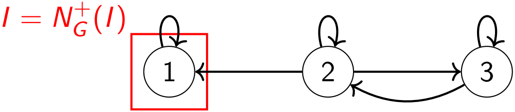


- Let $p_{(1\leftrightarrow 2)} : (x_1, x_2, x_3) \mapsto (x_2, x_1, x_3)$.
- $p_{(1\leftrightarrow 2)} \circ f(x) = (f_2(x), f_1(x), f_3(x))$



Even alphabets

- Let $p_{(1\leftrightarrow 2)} : (x_1, x_2, x_3) \mapsto (x_2, x_1, x_3)$.
- $p_{(1\leftrightarrow 2)} \circ f(x) = (f_2(x), f_1(x), f_3(x))$



- $p_{(1\leftrightarrow 2)}$ is the composition of $q^{n-2}\frac{q}{2}(q-1)$ transpositions:

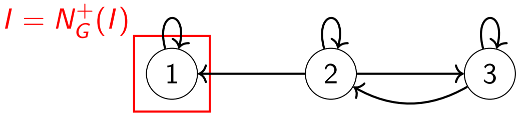
$$((x_1, x_2, x_3) \leftrightarrow (x_2, x_1, x_3)).$$

- For example:

$$((0,1,0) \leftrightarrow (1,0,0)).$$

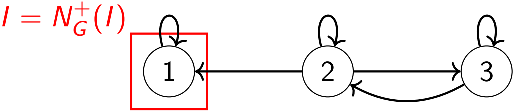
- So $p_{(1\leftrightarrow 2)}$ is even (if $n \geq 3$) and $p_{(1\leftrightarrow 2)} \circ f$ has the same parity as f.

Even alphabets

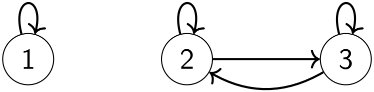


- So, we can consider that $I = N_G^+(I)$ and $\exists j \in V(G) \setminus N_G^-(I)$.
- For all $y \in [\![q]\!]^{n-|I|}$, let $h^{(y)}$ such that $f_I(x) = h^{(x_{V(G)\setminus I})}(x_I)$. $h^{(y)}$ is a permutation.
- Let $g : [\![q]\!]^n \to [\![q]\!]^n$ with $g_I(x) = (h^{(x_{V(G)\setminus I})})^{-1}(x_I)$ and $g_{V(G)\setminus I}(x) = \mathrm{id}$.
- Since I does not depend on v, $h^y = h^{y'}$ if y and y' only differ on j.
- Thus, $g(x)$ makes all permutations a multiple of q times.
- So, g is an even permutation.

Even alphabets



- So $f' = f \circ g$ as the same parity as f.
- $f'_{V(G)\setminus I} = f_{V(G)\setminus I}$ and for all $i \in I$, $f_i(x) = x_i$.



- f' is the product of the identity function $id : [\![q]\!]^{|I|} \to [\![q]\!]^{|I|}$ with $q^{|I|}$ fixed points and another permutation $f'_{V(G)\setminus I}$.
- So f' has a multiple of $q^{|I|}$ limit cycles.
- So f' is an even permutation.

Even alphabets

Lemma

Let $f \in F(G, q)$ be a permutation with q even and $n \geq 3$. Let $I \subseteq V(G)$. If $|I| = |N_G^+(I)|$ and $N_G^-(N_G^+(I)) \neq V(G)$. Then f has an even number of cycles.

Theorem

Let $f \in F(G, q)$ be a Hamiltonian automata network with q even and $n \geq 3$. Let $C_1, \ldots, C_m$ be the topologically ordered components of G. Then for all $1 \leq i \leq m$, $N_G^-(C_i) = C_1 \cup \cdots \cup C_i$.

- $f_{|C_1 \cup \ldots C_i}$ is Hamiltonian so suppose $f = f_{|C_1 \cup \ldots C_i}$.
- For contradiction, suppose $N_G^-(C_i) \neq C_1 \cup \cdots \cup C_i$.
- Take $I = C_i$. Clearly, $N_G^+(C_i) = C_i$ and $N_G^-(C_i) \neq V(G)$.
- By the Lemma, f has an even number of limit cycles. This is absurd (since it is Hamiltonian, it has one limit cycle).

Even alphabets

Lemma

Let $f \in F(G, q)$ be a permutation with q even and $n \geq 3$. Let $I \subseteq V(G)$. If $|I| = |N_G^+(I)|$ and $N_G^-(N_G^+(I)) \neq V(G)$. Then f has an even number of cycles.

Theorem

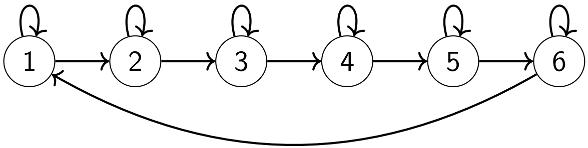
Let $f \in F(G, q)$ be a Hamiltonian automata network with q even and $n \geq 3$. Let $C_1, \ldots, C_m$ be the topologically ordered components of G. If $\{v\}$ is a feedback vertex set of C_i for some $1 \leq i \leq m$ then $N_G^-(v) = C_1 \cup \cdots \cup C_i$.

- Suppose $f = f_{|C_1 \cup \ldots C_i}$ and $N_G^-(v) \neq C_1 \cup \cdots \cup C_i$.
- $G[C_i \setminus \{v\}]$ is acyclic. Let u be the last vertex of a topological ordering of $G[C_i \setminus \{v\}]$.
- $I = \{u\}$, $N_G^+(I) = \{v\}$. By the lemma, f has an even number of limit cycles. This is absurd.

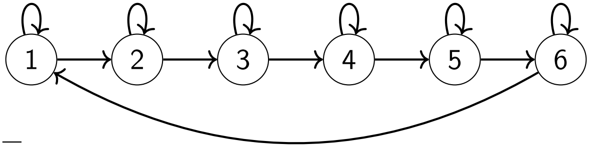
Partial results

Exemples of small degrees for even alphabets.

- We can prove that for $q = 2$, $n \geq 3$, the incoming degree of the interaction digraph is at least 3!
- But for $q = 4$ we can find proper graphs with incoming degree 2! (The biggest one we found is for $n = 6$.)
- It gives us, for $q = 2$, proper graphs with incoming degree 4.



Exemples of small degrees for even alphabets.



$\forall i, \ f_i(x) =$

i / si x_{i-1}	0	1	2	3
1	$(1 \leftrightarrow 2)x_1$	$(0 \leftrightarrow 1)x_1$	x_1	$(0 \leftrightarrow 2)x_1$
2	$(0 \leftrightarrow 3)x_2$	$(1 \leftrightarrow 3)x_2$	$(0 \leftrightarrow 1)x_2$	$(2 \leftrightarrow 3)x_2$
3	$(1 \leftrightarrow 2)x_3$	$(0 \leftrightarrow 1)x_3$	$(0 \leftrightarrow 2)x_3$	x_3
4	$(1 \leftrightarrow 2)x_4$	$(1 \leftrightarrow 2)x_4$	$(0 \leftrightarrow 2)x_4$	$(0 \leftrightarrow 1)x_4$
5	$(2 \leftrightarrow 3)x_5$	$(1 \leftrightarrow 2)x_5$	$(1 \leftrightarrow 3)x_5$	$(0 \leftrightarrow 3)x_5$
6	$(0 \leftrightarrow 2)x_6$	$(1 \leftrightarrow 2)x_6$	$(0 \leftrightarrow 1)x_6$	$(0 \leftrightarrow 1)x_6$

$000000 \to 030002 \to 103001 \to 010300 \to 002122 \to \dots$

Conclusion

Conclusion

In this talk, we have presented:

- Necessary conditions for the interaction graphs of Hamiltonian automata networks in general.
- More conditions when the alphabet is even.
- A construction of Hamiltonian automata network with an interaction digraph with a maximum in-degree of 2 working for all odd alphabet size.

Many questions remain open:

- Are the necessary conditions also sufficient?
- If $F(G, q)$ admits a Hamiltonian automata networks, does $F(G, q+2)$ too?