Complexity of the resolution of univariate polynomial equations over finite discrete dynamical systems

Seminar I3S
Thursday, November 14th

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A LITTLE CONTEXT

What is a finite discrete dynamical system (FDDS) ?

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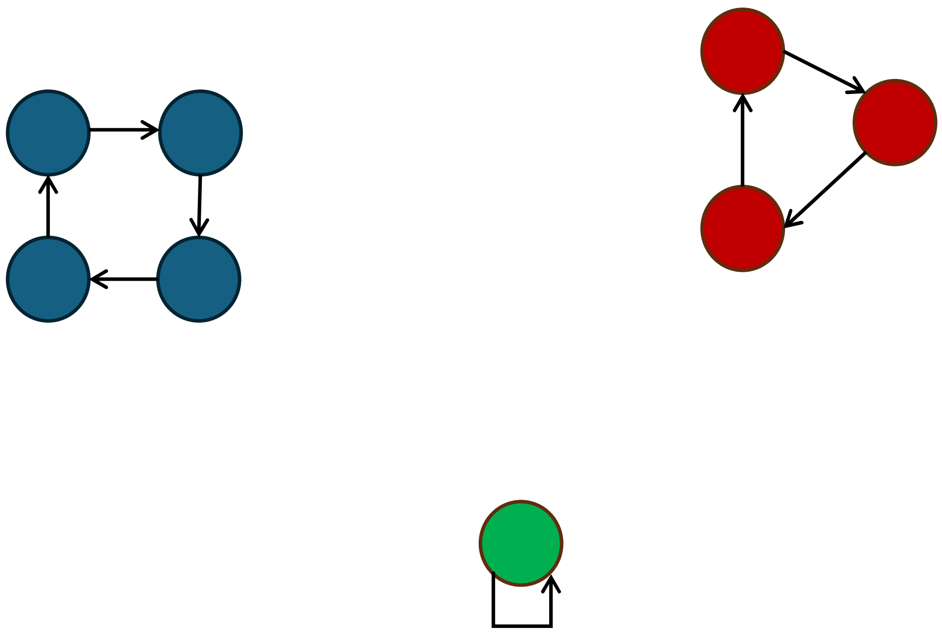

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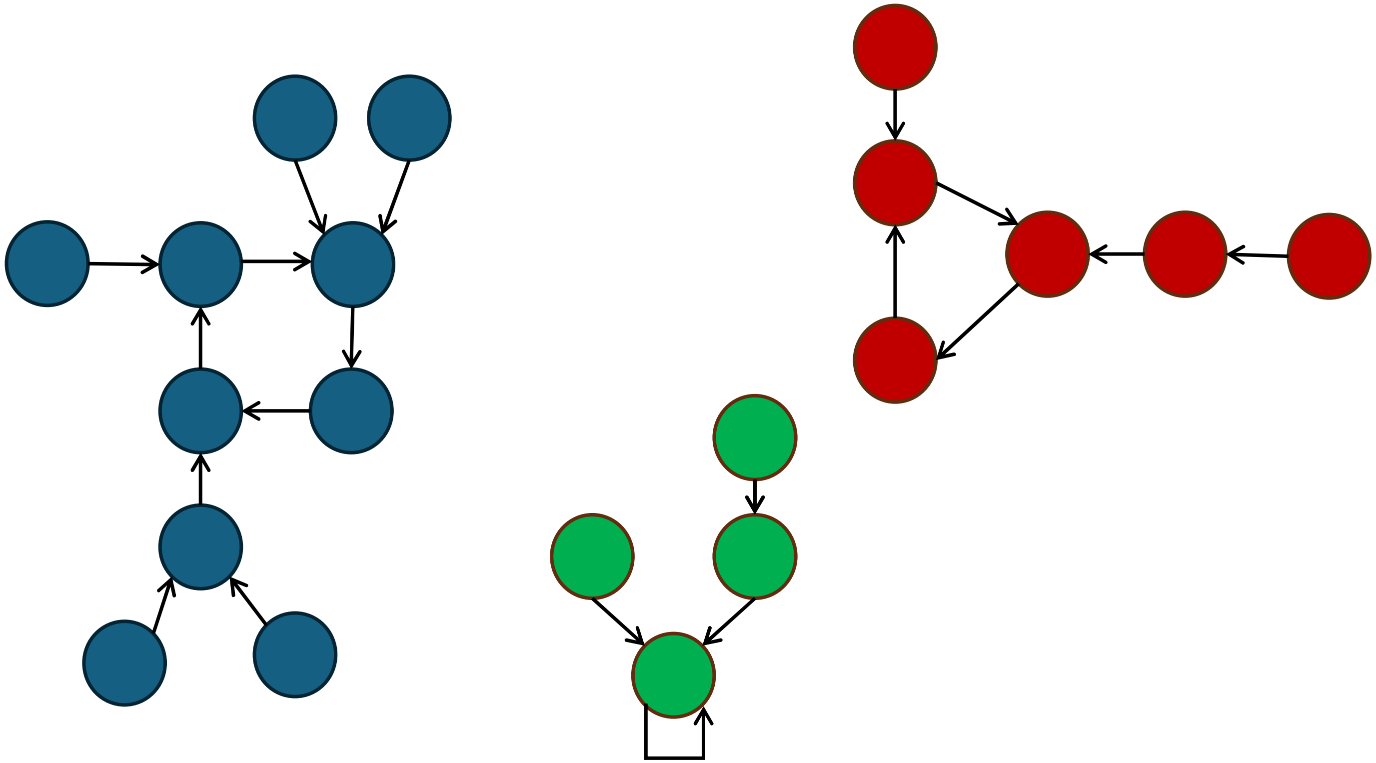
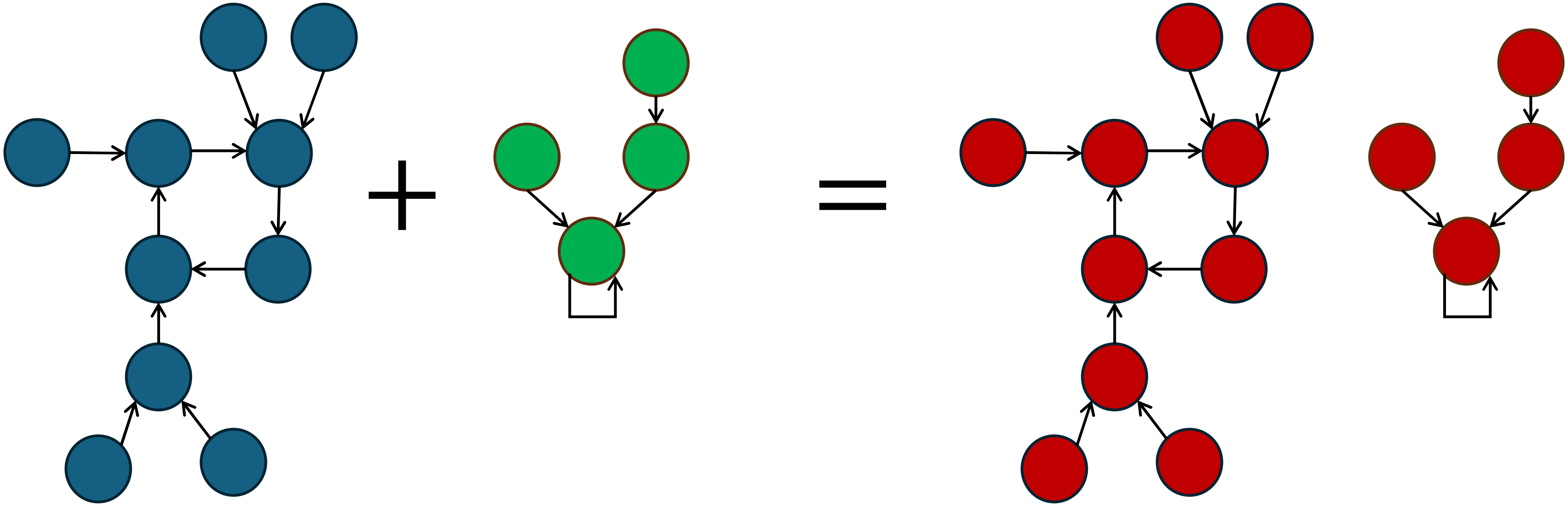


Operation : Sum



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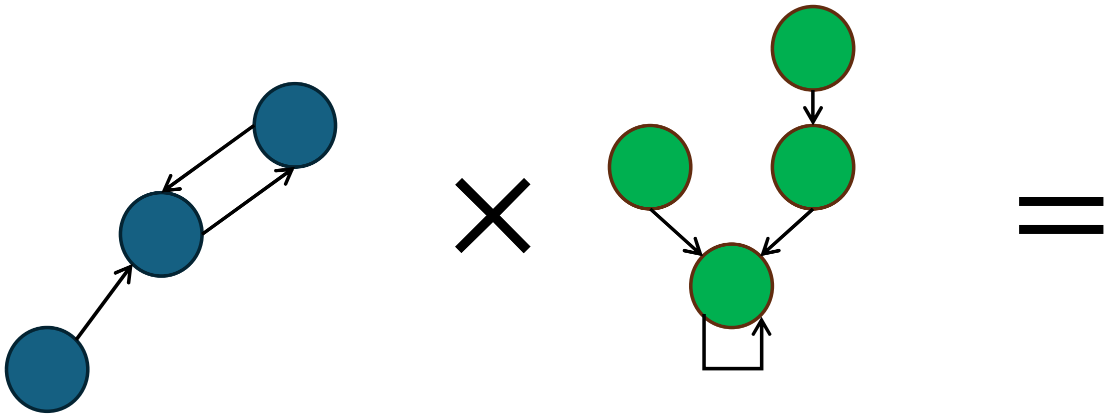

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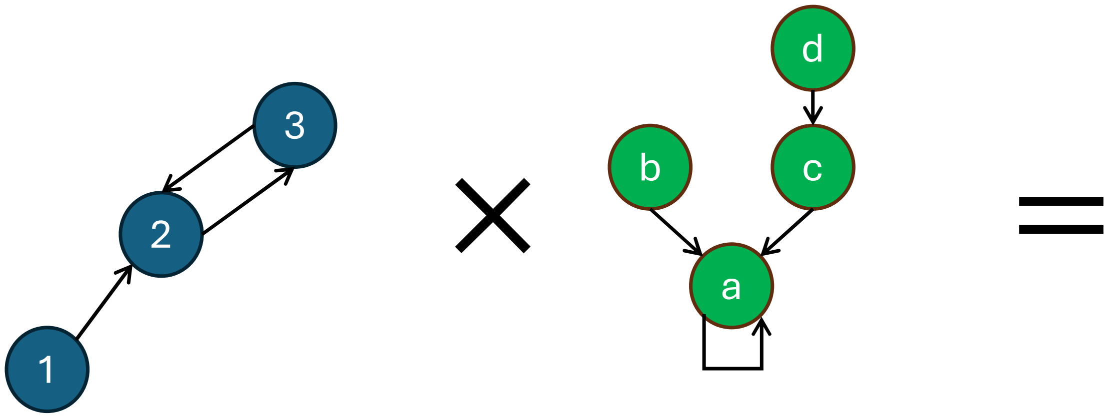

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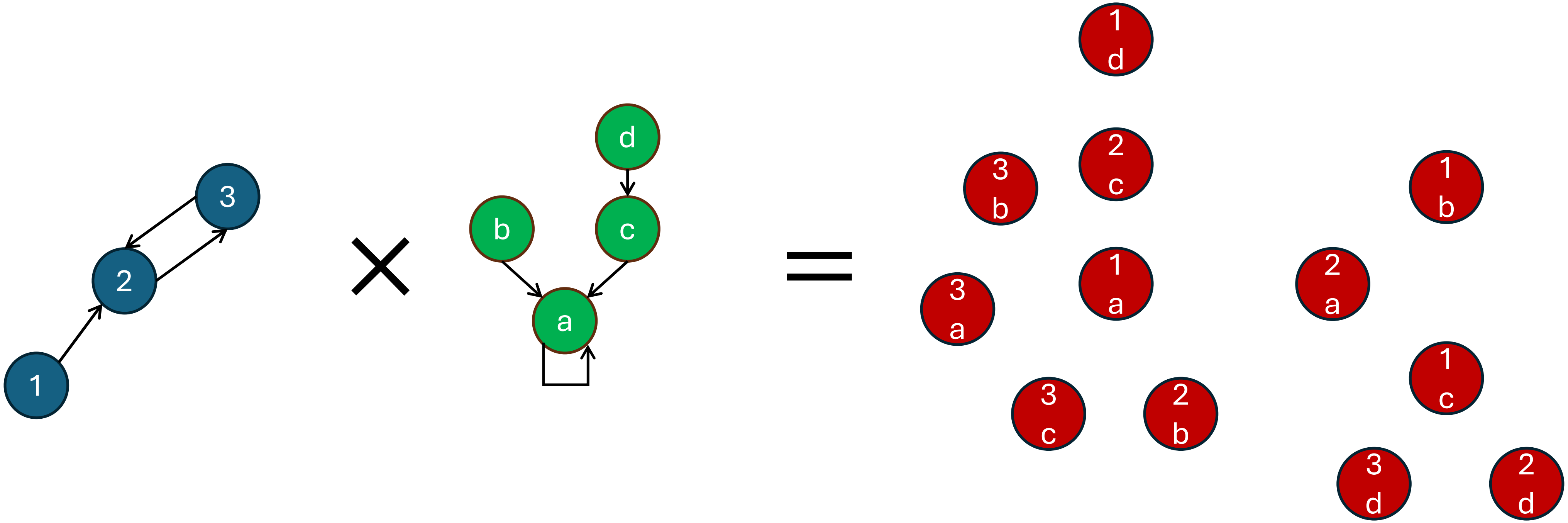

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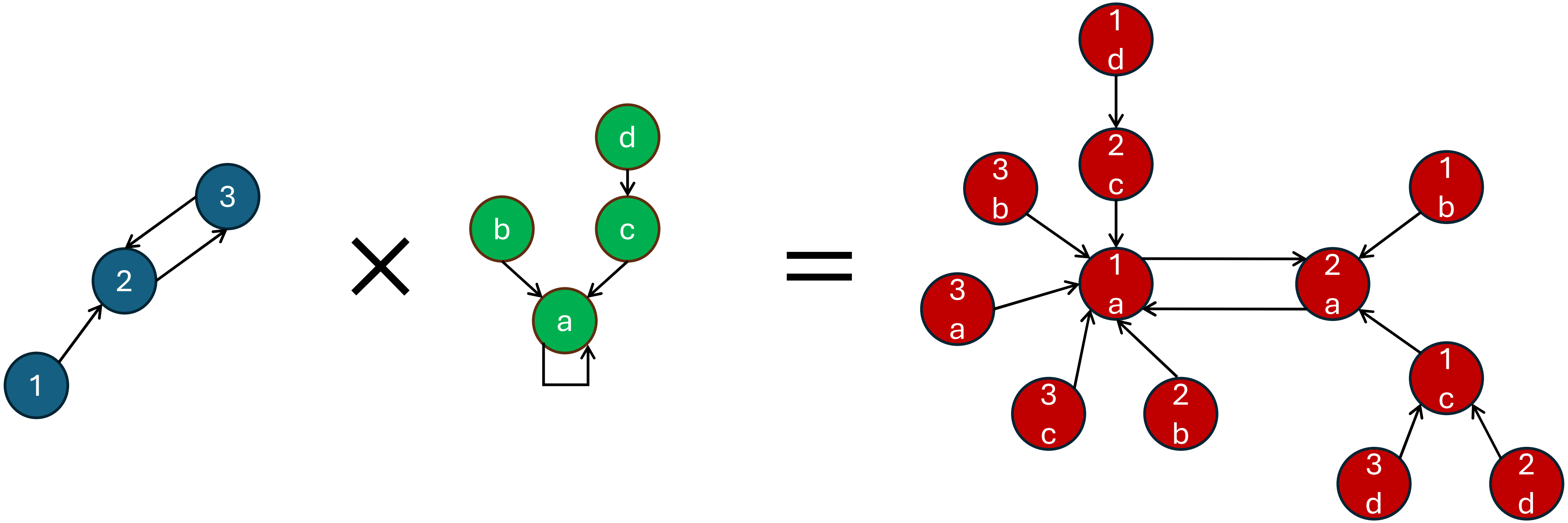

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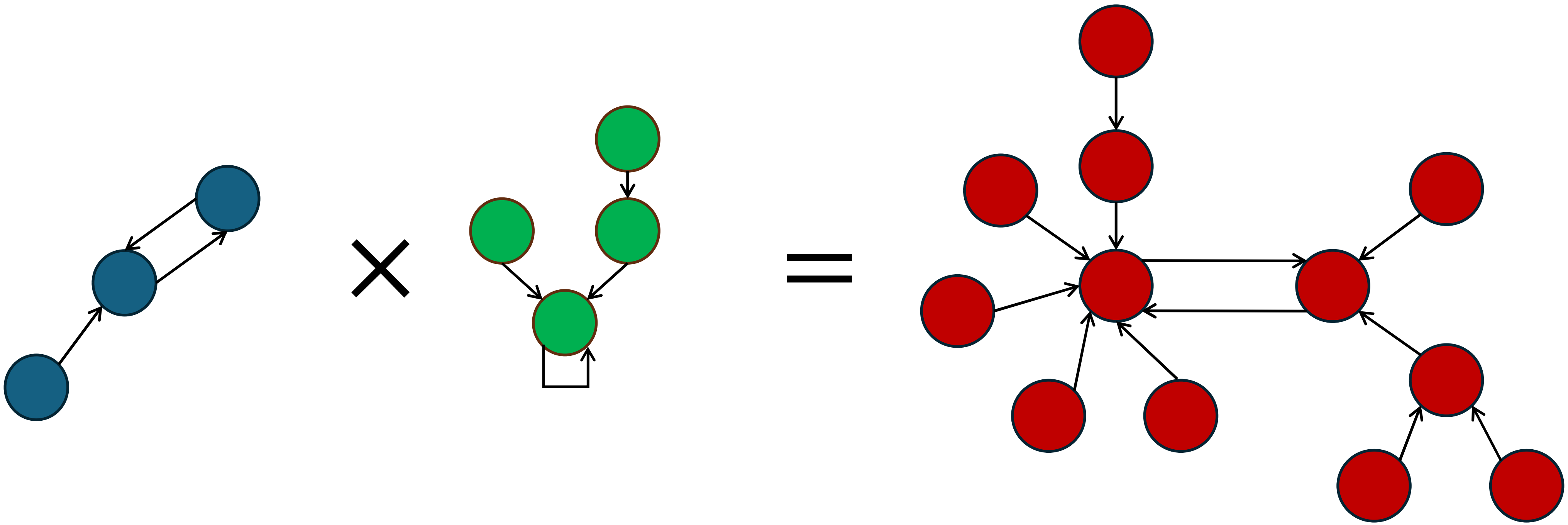

Property of product

The product of two connected FDDS A, B with cycle lengths a, b has $gcd(a, b)$ connected components with cycle length $lcm(a, b)$.

The Semiring $(\mathbf{D}, +, \times)$

The set **D** of FDDS **up to isomorphism** with the **alternative execution** as addition and **the synchronus execution** as multiplication is a commutative semiring.

Polynomial equations over FDDS

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- In P for the monomial univariate equations ($A\,X^k = B$) with connected result.

OUR NEW RESULTS

1. The behavior of transient nodes

First tool : Unroll

Main idea : Transform a FDDS into a forest where each tree is rooted in a periodic nodes and represents all possible paths in the FDDS between a node and its root.

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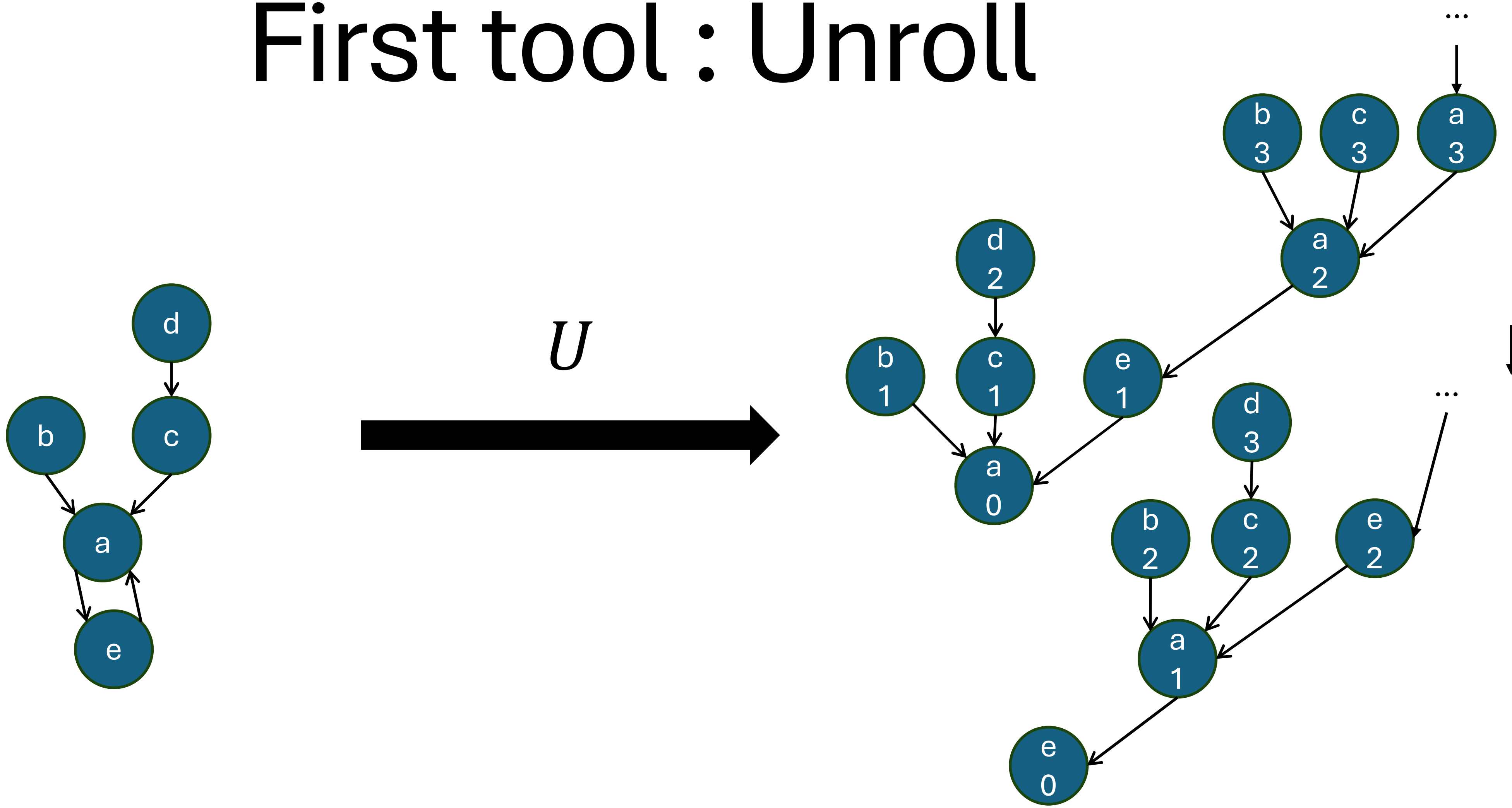
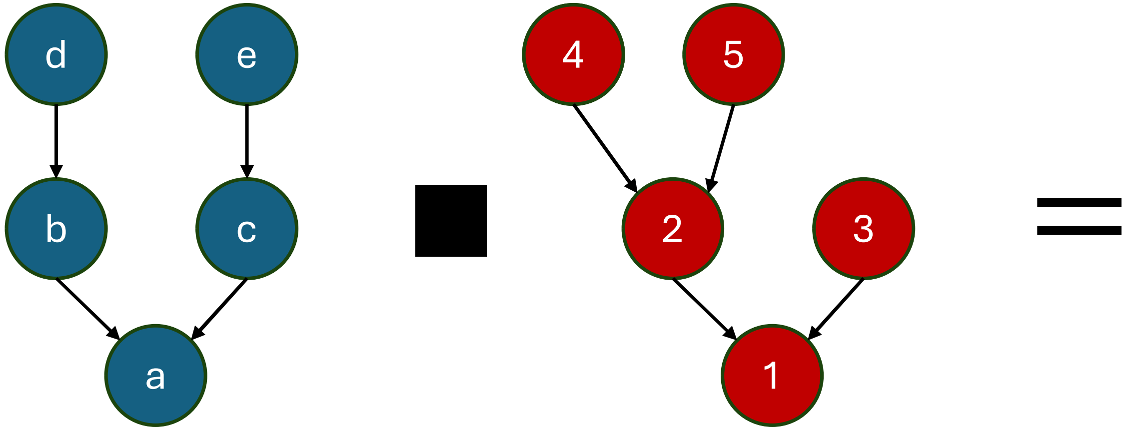


Tree product



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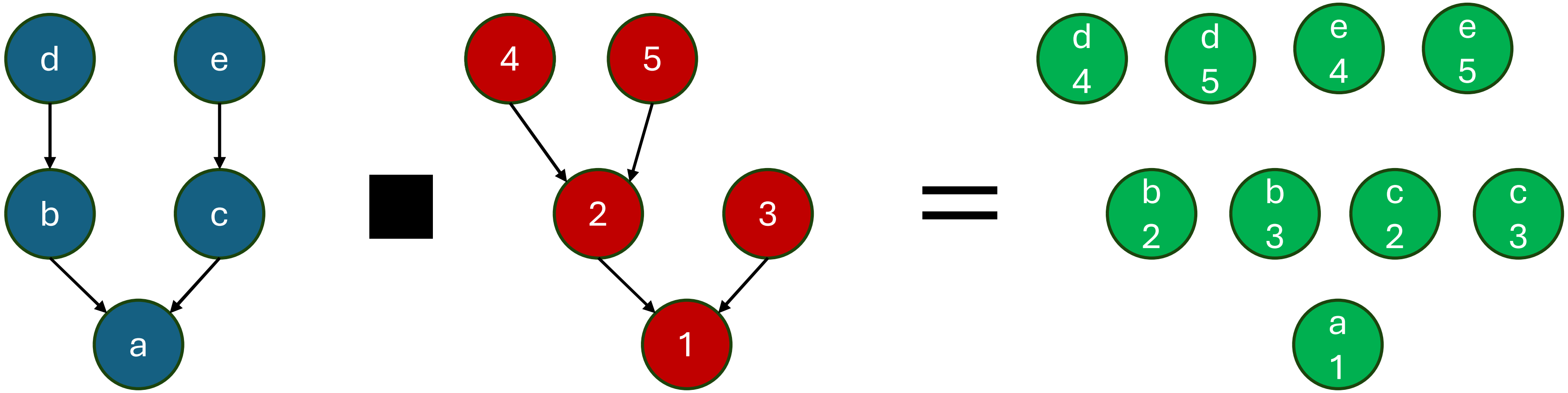

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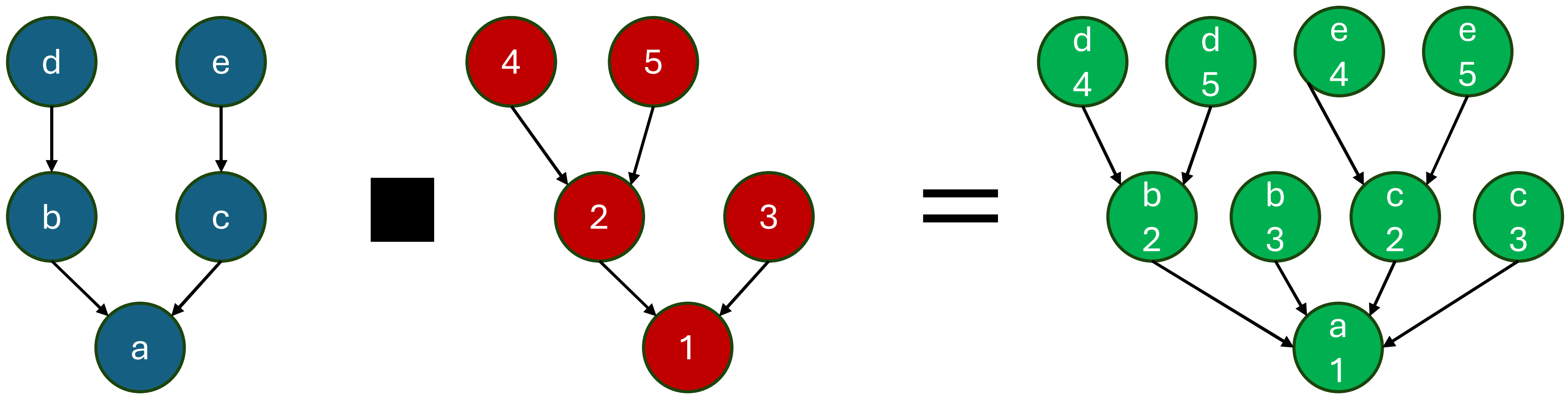

The Semiring $(U(\mathbf{D}), +, \blacksquare)$

The set U(**D**) of FDDS **up to isomorphism** with the **disjoint union** as addition and the **tree product** as multiplication is a commutative semiring.

Result over Unroll:

- U is a morphisme between **D** and U(**D**) ⇒ $U\left(\sum_{i=1}^{n} A_i X^i\right) = \sum_{i=1}^{n} U(A_i)\, U(X)^i$.

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- U is a morphisme between **D** and U(**D**) $\Rightarrow$ $U\left(\sum_{i=1}^{n} A_i X^i\right) = \sum_{i=1}^{n} U(A_i)\, U(X)^i$.
- There exists an order such that $A \leq B \Leftrightarrow A\,T \leq B\,T$ for all unroll tree A, B and T.

Second tool : Cut

Main idea : in a forest, consider only the nodes whose depth is less than a certain *n*.

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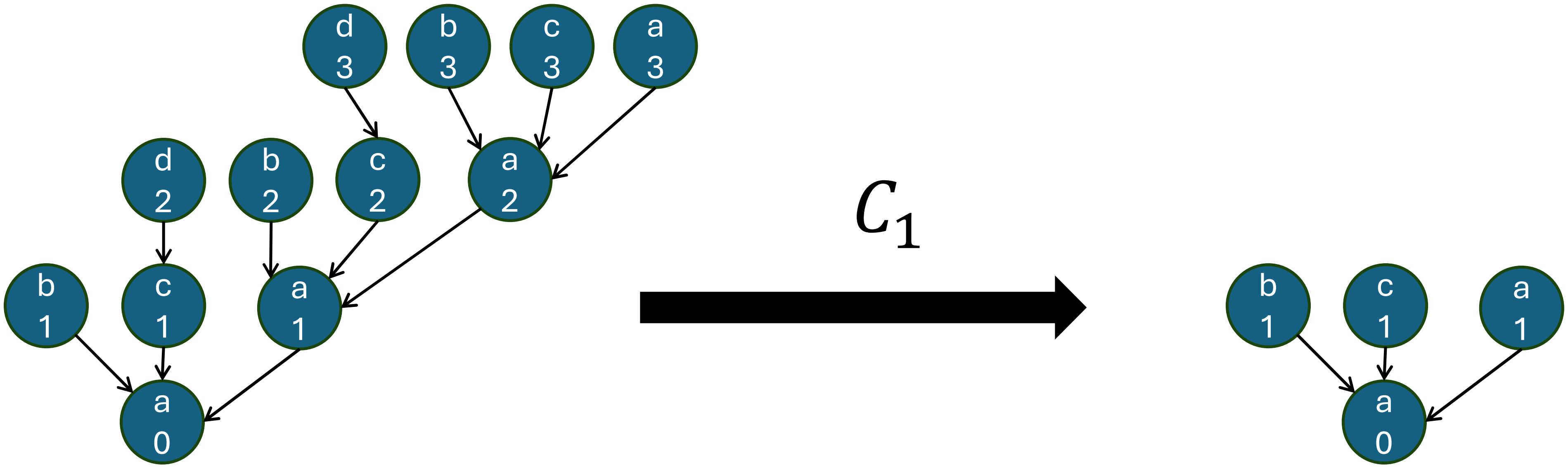

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- The previous order is such that $C_n(A) \leq C_n(B) \Leftrightarrow C_n(A)C_n(T) \leq C_n(B)C_n(T)$ for all unroll tree A, B and T.

Form of forests

Let $P = \sum_{i=1}^{m} A_i X^i$ be a polynomial over U_n and $X = \sum_{i=1}^{k} x_i$ with $x_i \leq \; x_{i+1}$. Then, there exists $i \; \in \{1, \dots, m\}$ such that :

1. The tree $\min\big(P(X)\big)$ is isomorphic to $a_1 {x_1}^i$
2. $\min(P(X) - P\big(\sum_{j=1}^{k-1} x_j)\big)$ is isomorphic to $a_1 {x_1}^{i-1} x_k$ for all $k \; \geq 2$

Where $a_1 = \; \min(A_i)$.

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- For all j, k, we have that :

$$a_1 {x_1}^i \quad \leq \quad \min\big(A_j\big)\, {x_1}^j$$

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- for all j, we have that $\min\big(X^{j-1} x_k\big) = \min\big(X^{j-1}\big)\, x_k = {x_1}^{j-1} x_k$
- So, $\min\big(A_j\big)\, {x_1}^{j-1} x_k \leq \min\big(A_j\big)\, y x_k \leq a y x_k$, for all $y \in X^{j-1}$, $a \in A_j$.

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    4. Return $P(Sol) = B$.

Consequences

1. We can solve in polynomial time $P(U(X)) = U(B)$ over Unroll.
2. All polynomial functions over Unroll are injective.

2. Polynomials over FDDS and algorithms

Step one: polynomials over permutations

Let k, n, m be three positive integer with $n \leq k$ and $X = X_1 + \ldots + X_k$ be an permutation such that $length(X_i) \leq length(X_{i+1})$ and $P(X) = \sum_{i=1}^{m} A_i X^i$ be a polynomial over permutation without constant term and with at least one cancellable coefficient.

Let B be a connected component of $P(X) - P(\sum_{i=1}^{n-1} X_i)$ with minimal cycle length and $p = length(B)$.

Then $length(X_n) = p$.

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3. $X \subseteq X^i$ for all integer i greater than *1* $\Rightarrow X = XC_1 \subseteq P(X)$.

Step one: polynomials over permutations

$$(C_3+C_2)\, X + (C_4 + \; C_1)X^2 = 4\, C_2 + 6\, C_3 + 42\, C_4 + 4\, C_6 + 18\, C_{12}$$

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$$P(X) - P(C_2\,) = 6\, C_3 + 38\, C_4 + 3\, C_6 + 4\, C_{12}$$

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$$P(X) - P(C_2) = 6\,C_3 + 28\,C_4 + 3\,C_6 + 18\,C_{12}$$

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$$Sol = C_2 + C_3 + C_4$$

$$P(X) - P(C_2 + C_3 + C_4) = 0$$

Step one: polynomials over permutations

Let P be a polynomial over permutations. If at least one non-constant coefficient is cancellable then the polynomial is injective and we can solve $P(X) = B$ in polynomial time.

Step two: root

Let a, b be two integers and A, B be two connected FDDS such that $length(A) = a$ and $length(B) = b$.

Let $t_A = \min(\,U(A))$ and $t_B = \min\big(U(B)\big)$. We define the orders:

- $A \;\leq_c\; B\ if\ and\ only\ if\ \ a < b\ or\ (a = b\ and\ t_A\ \leq\ t_B)$

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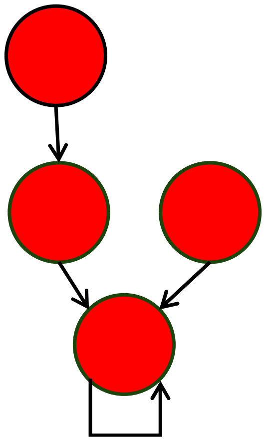
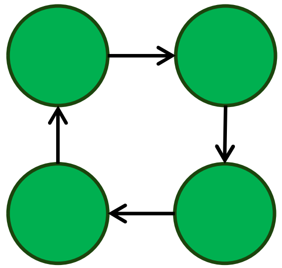


$\leq_c$



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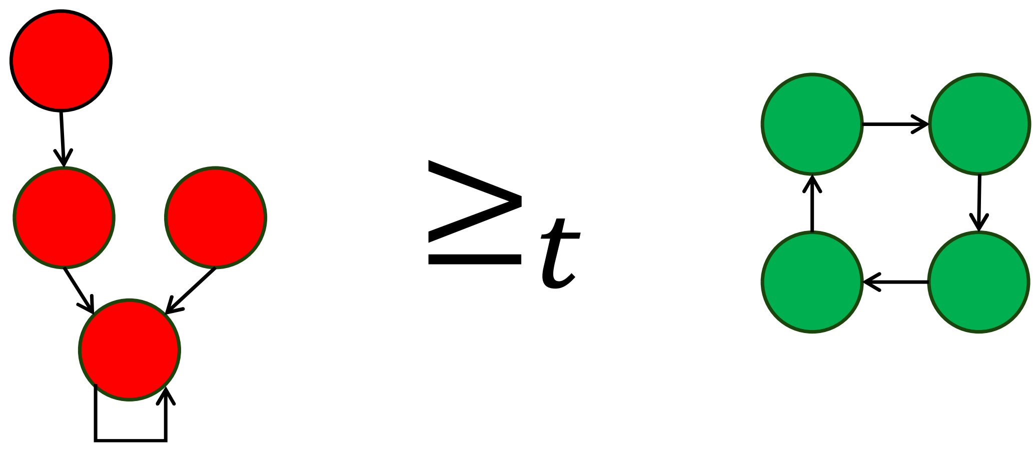

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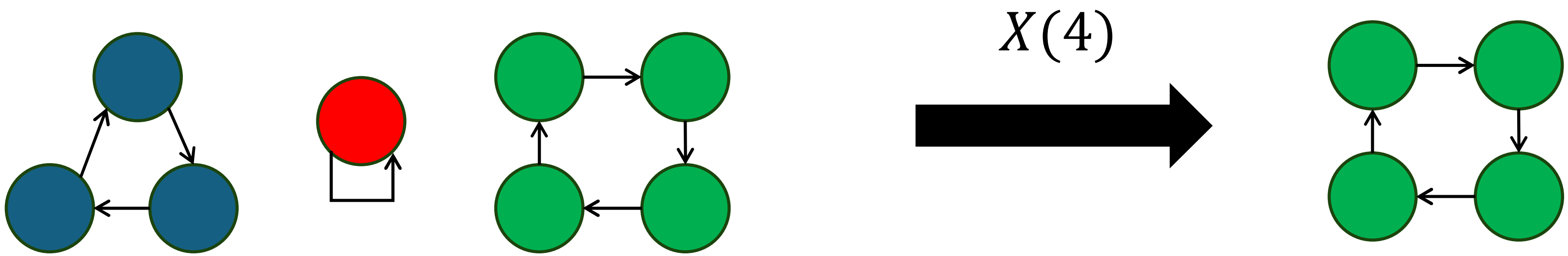

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Step two: root

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Then B is a connected component of ${X_l}^{k-1} X_n$ with $X_l = \min(X\{p\})$ according to $\leq_t$.

Step two: root

Idea of proof :

1. The connected component B is the product of $m \; - 1$ elements of $X\{p\}$ and an element of $(X^m - \left(\sum_{i=1}^{n-1} X_i\right)^{ms})(p)$

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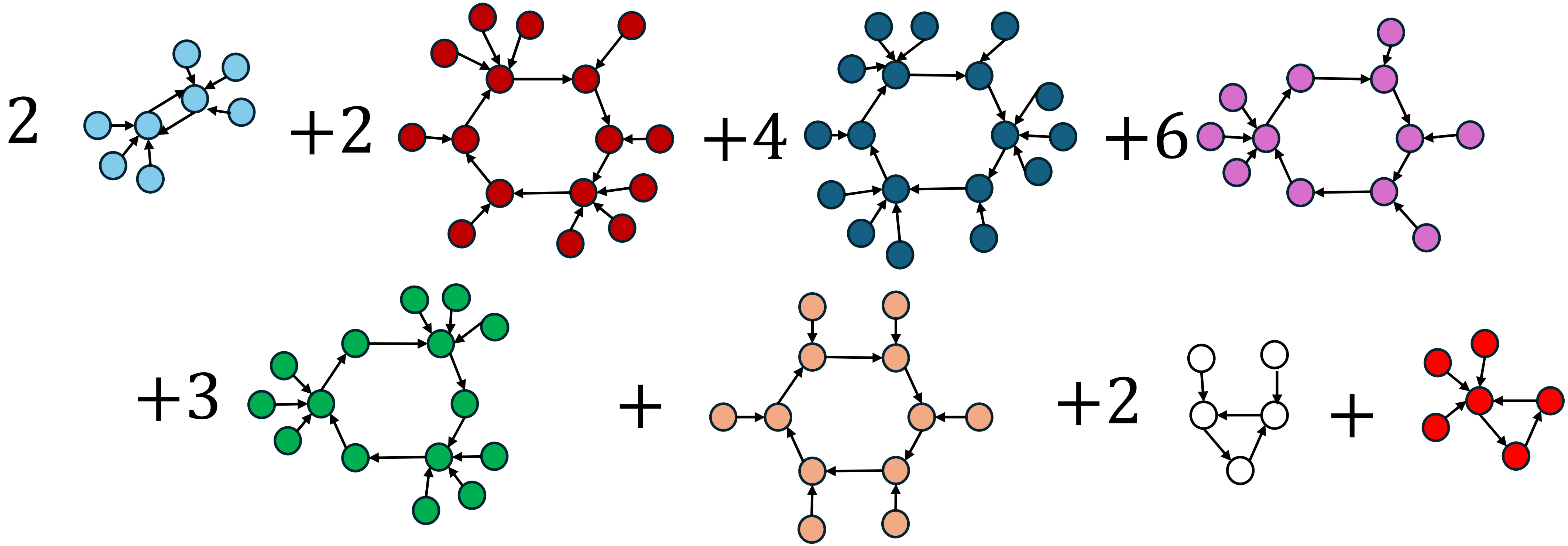
2. The minimal unroll tree of B is the product of the minimal unroll tree in $X\{p\}$ raised to the power $m-1$ and the minimal unroll tree in $(X - \sum_{i=1}^{n-1} X_i)(p)$.

Step two: root
Example: square root



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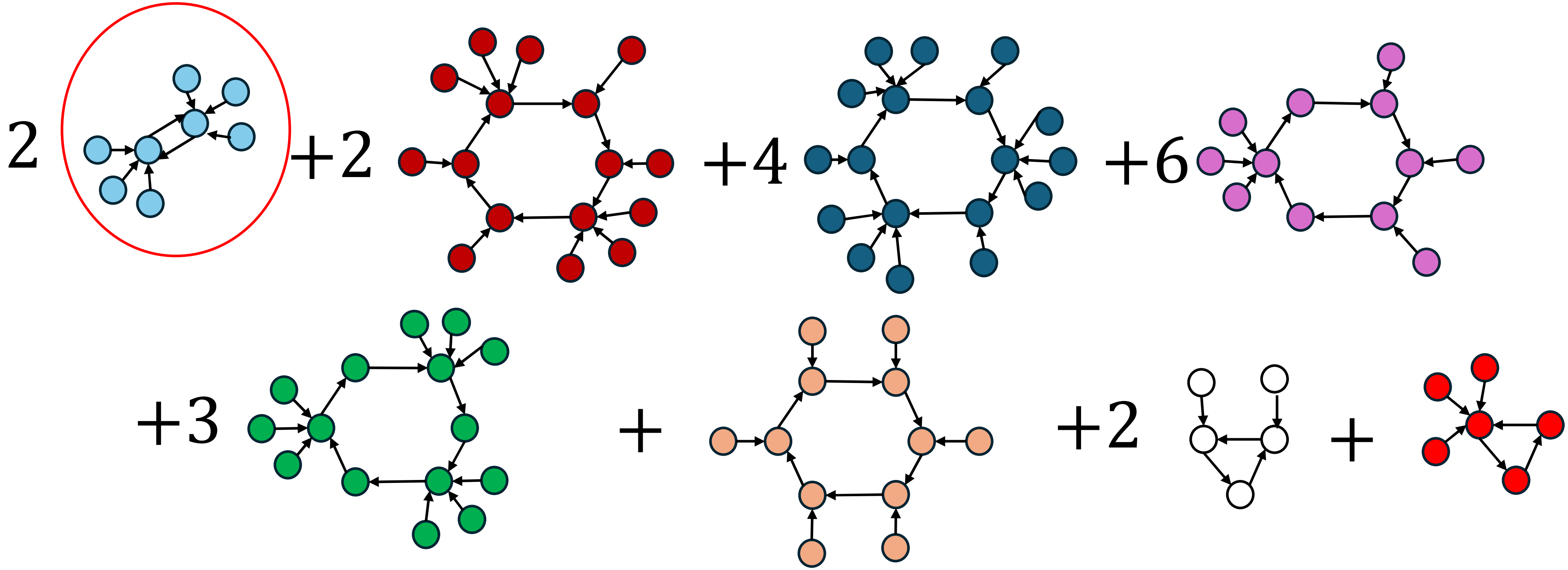
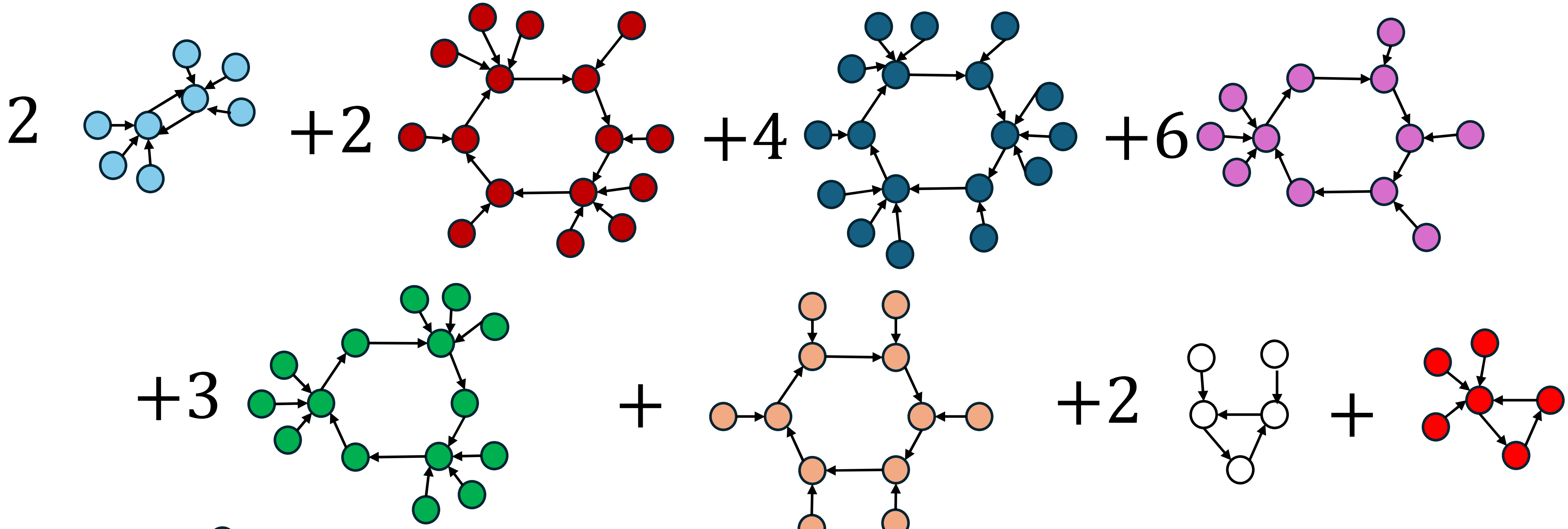
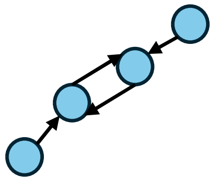


Step two: root
Example: square root



$X =$ 

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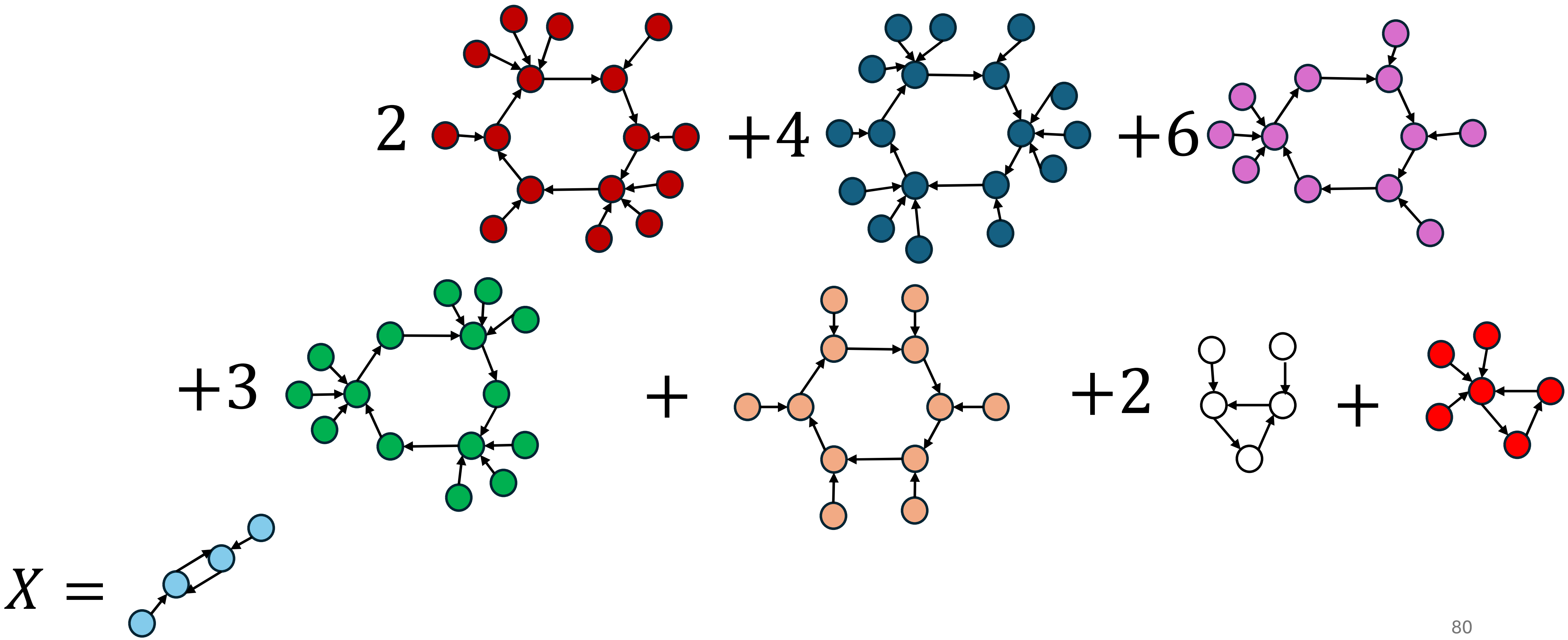

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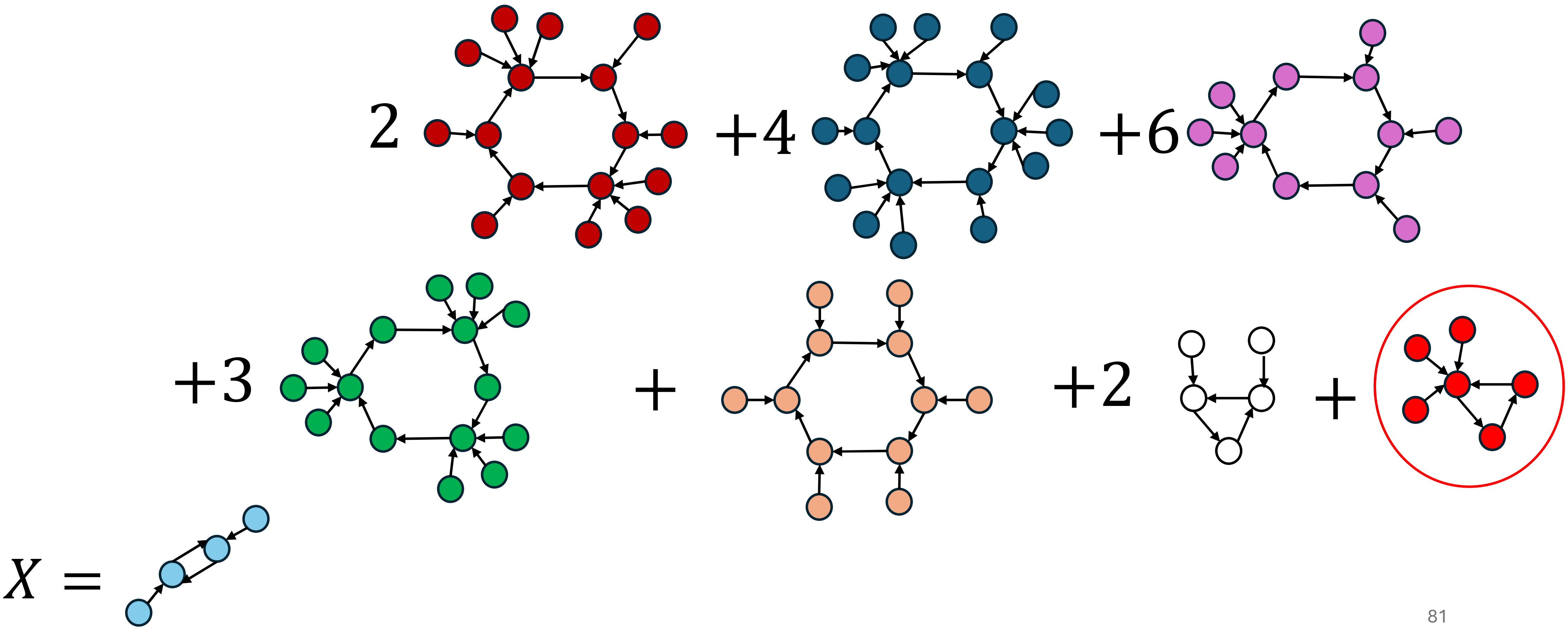

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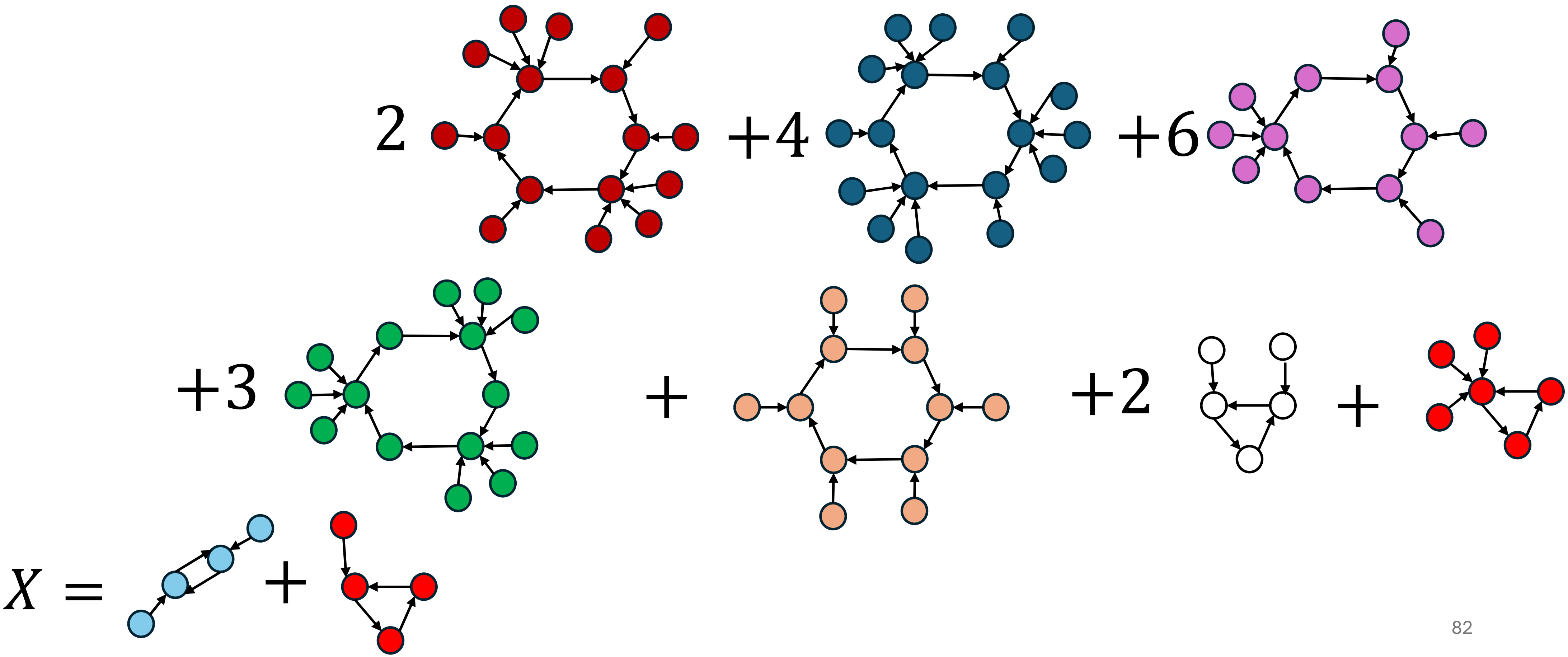

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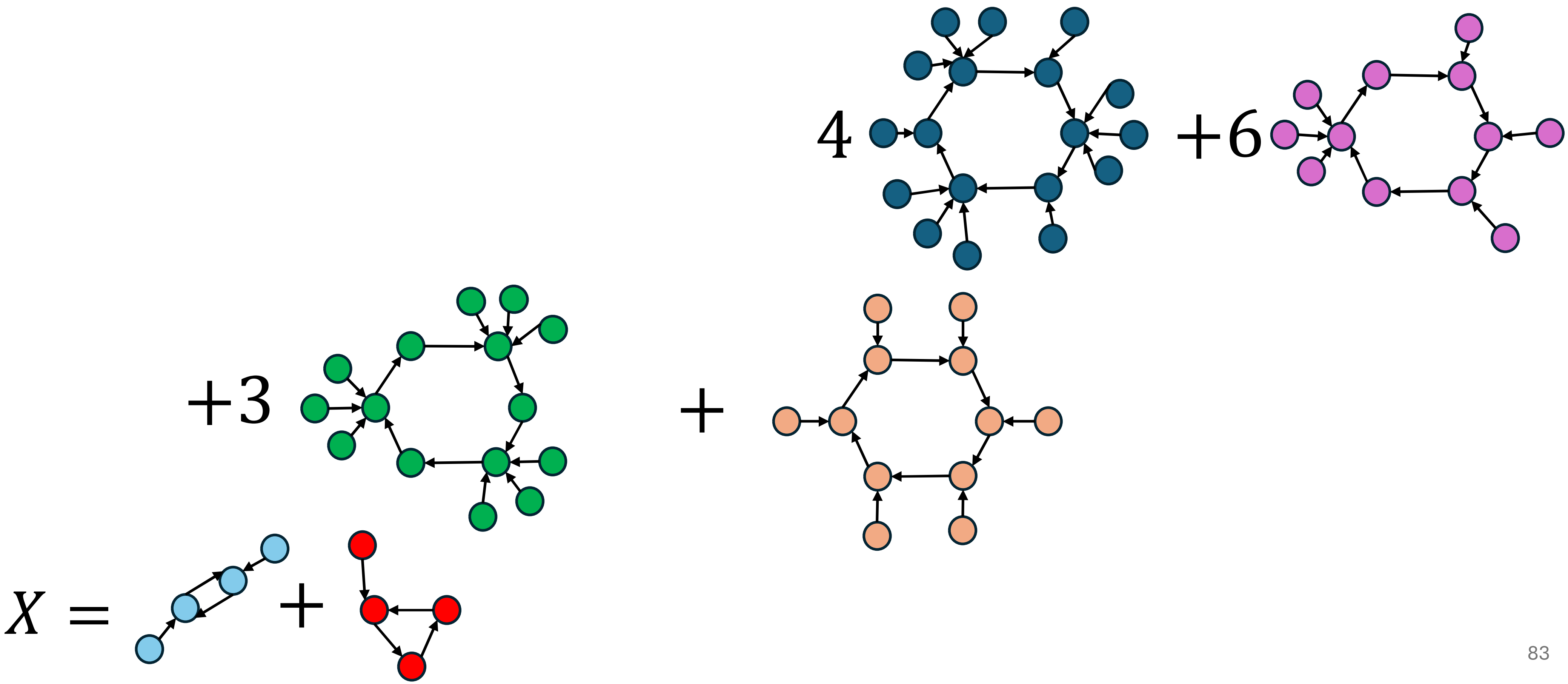

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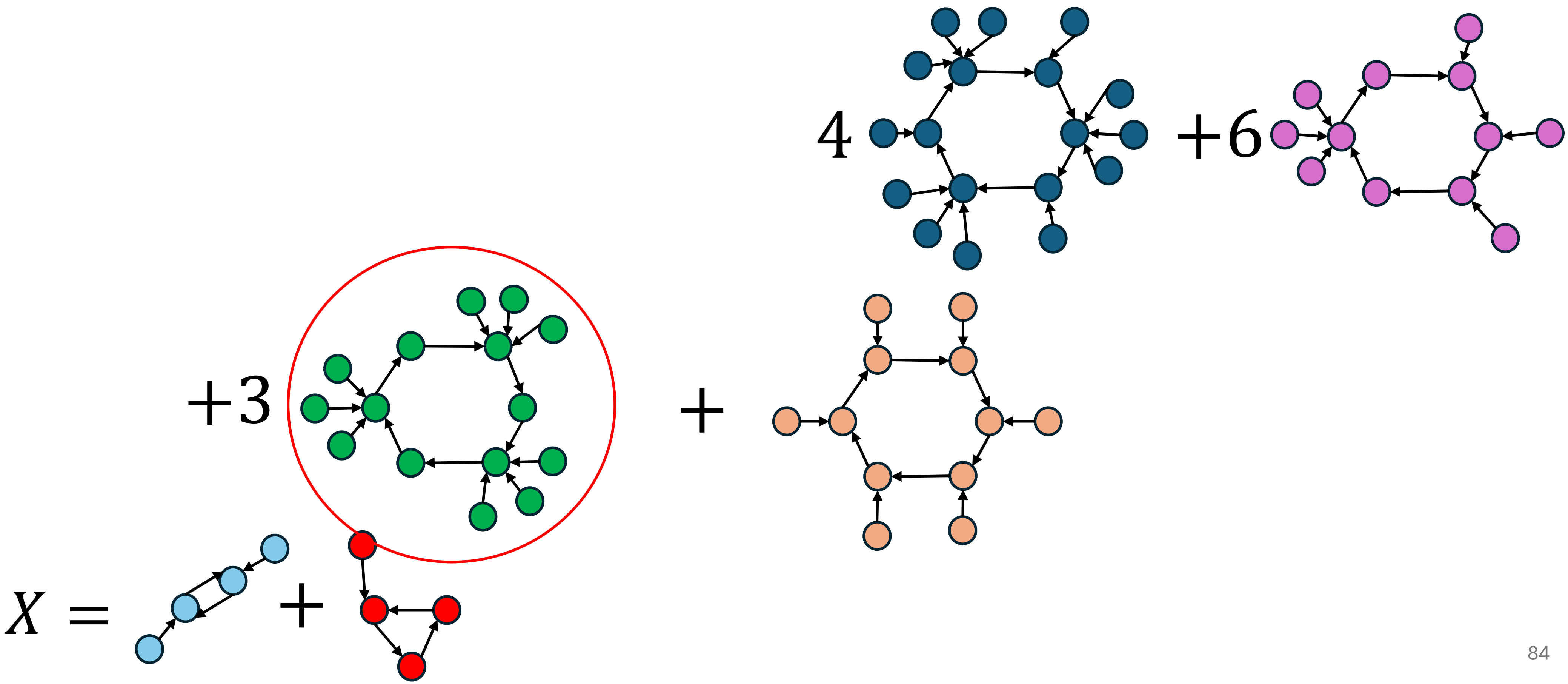

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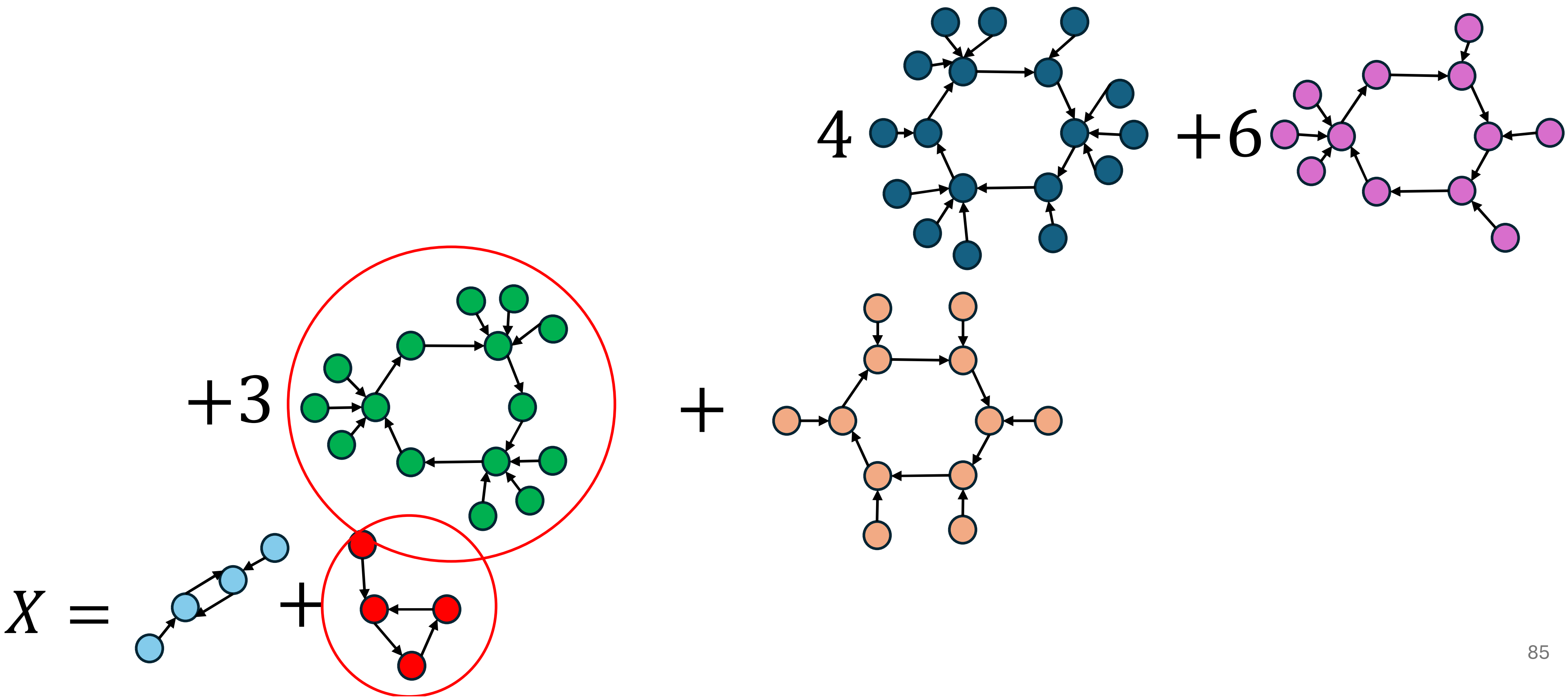

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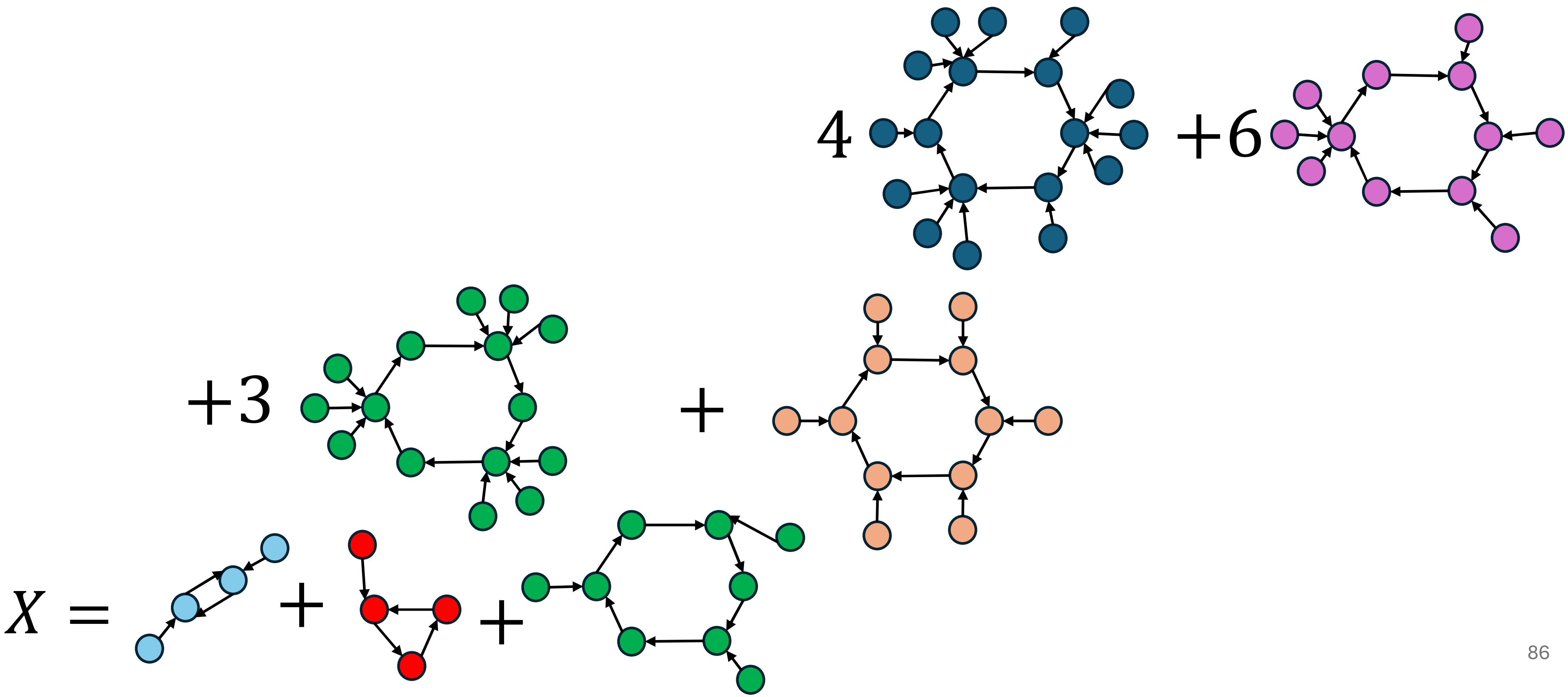

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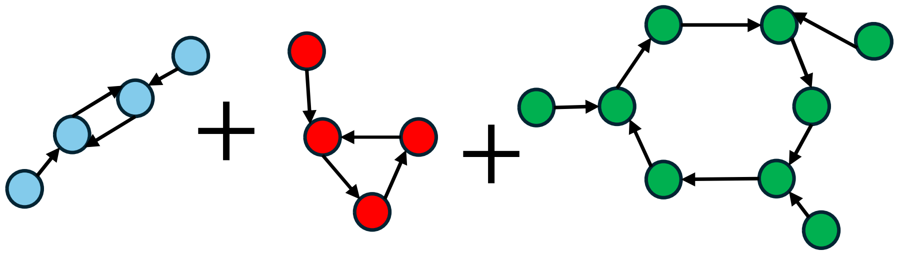
$X =$ 

Step two: root

$\sqrt[m]{\cdot}$ is injective and we can compute it in polynomial time.

Step three: general FDDS

Let k, n, m be three positive integer with $n \leq k$ and $X = X_1 + \ldots + X_k$ be an FDDS such that $X_i \leq_c X_{i+1}$ and $P = \sum_{i=1}^{m} A_i X^i$ be a polynomial without constant term and with at least one cancellable coefficient. Let B be a connected component of $\min(P(X) - P(\sum_{i=1}^{n-1} X_i))$ according to $\leq_c$ and $p = length(B)$.

Then $length(X_n) = p$.

Step three: general FDDS

- Since $length(X_n)$ is the smallest cycle length in $X - \sum_{i=1}^{n-1} X_i$, we have that $length(X_n) \leq length(B)$.

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- Since P contains a cancellable coefficient, there exists a connected coefficient B' in $P(X) - P(\sum_{i=1}^{n-1} X_i)$ such that $length(B') = length(X_n) \Rightarrow length(X_n) \geq length(\mathrm{B})$.

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   3. Roll the minimum tree of this solution with period p,
   4. Add the resulting component to Sol,
   5. Repeat until $P(Sol) \not\subset B$,
   6. Return $P(Sol) = B$.

Thanks for your attention

References

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