# Finite Dynamical Systems and Divisor of Sum of Cycles

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# Deterministic Finite Dynamical Systems

**Definition:**

- A deterministic, finite, discrete-time dynamical system is a function mapping a finite set of states to itself.
- Represented as a *functional digraph* (up to an isomorphism): a finite directed graph where each vertex has a unique out-neighbor.

**Some references:**

- Alberto Dennunzio, Valentina Dorigatti, Enrico Formenti, Luca Manzoni, and Antonio E Porreca
- François Doré, Kévin Perrot, Antonio E. Porreca, Sara Riva and Marius Rolland
- Émile Naquin, Maximilien Gadouleau
- Florian Bridoux, Christophe Crespelle, Thi Ha Duong Phan, and Adrien Richard

# Algebraic Operations on Functional Digraphs

Digraph $A$: vertex set $V(A)$, edge set $E(A)$.

**Addition ($A + B$):**

- Disjoint union of functional digraphs $A$ and $B$.

**Multiplication ($AB$):**

- Direct product of $A$ and $B$.
- Vertex set $V(AB)$ is the Cartesian product $V(A) \times V(B)$.
- Edge from $(x, y) \to (x', y')$, if : $x \to x'$ in $A$ and $y \to y'$ in $B$.

**Semiring Structure:**

- The set of functional digraphs with these operations forms a *semiring*.

# Division Problem in Functional Digraphs

- **Multiplicative Structure**
- **Division Problem:**
  - Given functional digraphs $A$ and $B$, does $A$ divide $B$?
  - Exists a solution $X$ to $AX = B$?
- **Complexity:**
  - Problem is in NP.
  - General case remains open.
  - Better upper bounds under certain conditions.

# Structure of Functional digraph

- **Definition:** A functional digraph $A$ has every vertex with exactly one outgoing edge.
- **Components:**
  - **Cyclic Part:** Collection of disjoint cycles.
- **Transient Part:**
  - Remaining structure after removing cycles.
  - Disjoint union of out-trees.

**We focus on the case: each graph be a collection of disjoint cycles - called Sum of cycles**

# Divisors of Graphs as Sum of Cycles

- **Objective:**
  - Investigate divisors of graphs that can be represented as a sum of cycles.
- **Focus Areas:**
  - **Cycle Lengths as Powers of a Prime:**
    - Initial focus on cycles where lengths are powers of the same prime number.
  - **Generalization Using Recurrence:**
    - Employ a recurrence method.
    - Based on the number of distinct prime factors in cycles.

Let $X$ and $Y$ be two sums of cycles. We define two operators, called addition and product, as follows:
For sums of cycles:

$$X = x_1 C_1 + x_2 C_2 + \cdots + x_m C_m,$$

$$Y = y_1 C_1 + y_2 C_2 + \cdots + y_k C_k,$$

**Addition**: The sum $Z = X + Y$ is defined by

$$Z = \sum_{i=1}^{m} z_i C_i,$$

where $z_i = x_i + y_i$.

# Multiplication of Cycles

**Multiplication**: The product $Z = X \cdot Y$ is defined by

$$Z = \sum_{0 \le i \le m,\, 0 \le j \le k} x_i y_j C_i C_j,$$

where

$$C_i C_j = \gcd(i, j) C_{\mathrm{lcm}(i,j)}.$$

We denote by $n(X)$ the size of $X$, that means

$$n(X) = x_1 + 2x_2 + \cdots + m x_m.$$

# Cycles with One Prime Number

Let $X$ be a sum of cycles where the lengths are powers of the same prime. We write $X$ as:

$$X = x_1 C_1 + x_p C_p + x_{p^2} C_{p^2} + \cdots + x_{p^m} C_{p^m}.$$

The set $\mathbb{N}[p]$, with the operations of addition and multiplication, forms a semiring over $\mathbb{N}$.
Extension: the set $\mathbb{Q}[p]$, with the operations of addition and multiplication, forms a ring over $\mathbb{Q}$.

# Algorithm to Check Divisors

**Introduction to Divisors of Cycles**

- Let $B$ be a sum of cycles, each of length being a power of the same prime $p$:

$$B = b_1 C_1 + b_{p^{\alpha_1}} C_{p^{\alpha_1}} + b_{p^{\alpha_2}} C_{p^{\alpha_2}} + \ldots + b_{p^{\alpha_k}} C_{p^{\alpha_k}}$$

- Conditions:
  - $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k$
  - $b_{p^{\alpha_i}} > 0$ for $i = 1, 2, \ldots, k$
  - $b_1 \geq 0$

**Key Question**: What form must a divisor $A$ of $B$ take?

# Form of Divisors

**Lemma on Forms of $A$ and $X$**

- If $B = A \cdot X$, then:

$$A = A_* + \sum_{i=1}^{k} a_{p^{\alpha_i}} C_{p^{\alpha_i}}, \quad X = X_* + \sum_{i=1}^{k} x_{p^{\alpha_i}} C_{p^{\alpha_i}}$$

- Where:

$$A_* = \sum_{j=0}^{\alpha_1 - 1} a_{p^j} C_{p^j}, \quad X_* = \sum_{j=0}^{\alpha_1 - 1} x_{p^j} C_{p^j}$$

- $a_{p^j}, x_{p^j}$ are non-negative integers.

**Two Cases**:

- If $b_1 \neq 0$:

$$A_* = a_1 C_1, \quad X_* = x_1 C_1 \text{ with } x_1 = \frac{b_1}{a_1}$$

- If $b_1 = 0$:

$$A_* = 0 \text{ or } X_* = 0$$

**Note**: $X$ is called a quotient of $\frac{B}{A}$, different quotients $X$ may exist.

# Lemma: Form of $A$ and $X$

For all $1 \leq l \leq k$:

$$(a_* + \sum_{i=1}^{k} a_{p^{\alpha_i}} C_{p^{\alpha_i}})(x_* + \sum_{i=1}^{l} x_{p^{\alpha_i}} C_{p^{\alpha_i}}) \quad = b_1 + \sum_{i=1}^{l} b_{p^{\alpha_i}} C_{p^{\alpha_i}}.$$

**Special Case:** If $b_1 \neq 0$, then $a_* = a_1$ and $x_* = x_1$.

# Computing $x_{p^{\alpha_i}}$

**In all cases:**

$$
\begin{aligned}
a_* x_* &= b_1 \\
x_{p^{\alpha_1}} &= \frac{1}{p^{\alpha_1}} \left( \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}}} - x_* \right) \\
x_{p^{\alpha_2}} &= \frac{1}{p^{\alpha_2}} \left( \frac{b_1 + \sum_{i=1}^{2} p^{\alpha_i} b_{p^{\alpha_i}}}{a_* + \sum_{i=1}^{2} p^{\alpha_i} a_{p^{\alpha_i}}} - \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}}} \right) \\
\ldots & \\
x_{p^{\alpha_k}} &= \frac{1}{p^{\alpha_k}} \left( \frac{b_1 + \sum_{i=1}^{k} p^{\alpha_i} b_{p^{\alpha_i}}}{a_* + \sum_{i=1}^{k} p^{\alpha_i} a_{p^{\alpha_i}}} - \frac{b_1 + \sum_{i=1}^{k-1} p^{\alpha_i} b_{p^{\alpha_i}}}{a_* + \sum_{i=1}^{k-1} p^{\alpha_i} a_{p^{\alpha_i}}} \right).
\end{aligned}
$$

# Uniqueness and Non-negativity of $x_{p^{\alpha_i}}$ (for $i = 2, \ldots, k$)

- $x_{p^{\alpha_i}} = \frac{Q_i - Q_{i-1}}{p^{\alpha_i}}$
  uniquely determined and non-negative iff: condition (*):

$$Q_{i-1} = \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}} + \ldots + p^{\alpha_{i-1}} b_{p^{\alpha_{i-1}}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}} + \ldots + p^{\alpha_{i-1}} a_{p^{\alpha_{i-1}}}} \equiv \frac{n(B)}{n(A)} \mod p^{\alpha_i}$$

$$Q_i \geq Q_{i-1}$$

- $Q_1, Q_2, \ldots, Q_k$ satisfying (*): suitable sequence.
- **Other Variables:**
  - Variables $x_j$ with $j \leq p^{\alpha_1}$ depend on $a_*, a_{p^{\alpha_1}}, b_1, b_{p^{\alpha_1}}$.
  - There may be no solution, one solution, or multiple solutions for these variables.

# Number of solutions

**Solve the First Equation**

- If $a_* \neq 0$, then the solution is unique.
- If $a_* = 0$, then all $x_{p^{\alpha_i}}$ for $i \geq 2$ are already determined. For the first variables, solve the equation:

$$x_1 + px_p + p^2 x_{p^2} + \ldots + p^{\alpha_1 - 1} x_{p^{\alpha_1 - 1}} + x_{p^{\alpha_1}} p^{\alpha_1} = \frac{b_{p^{\alpha_1}}}{a_{p^{\alpha_1}}}.$$

- There are $s\left(\frac{b_{p^{\alpha_1}}}{a_{p^{\alpha_1}}}, p, \alpha_1\right)$ solutions, where $s(n, p, k)$ is the number of solutions of the equation:

$$x_1 + px_p + p^2 x_{p^2} + \ldots + p^{k-1} x_{p^{k-1}} + x_{p^k} p^k = n.$$

# Theorem: Algorithm for Divisibility in Case of single prime

Theorem

*Let* $B = b_1 C_1 + b_{p^{\alpha_1}} C_{p^{\alpha_1}} + b_{p^{\alpha_2}} C_{p^{\alpha_2}} + \ldots + b_{p^{\alpha_k}} C_{p^{\alpha_k}}$.
*Let* $A$ *be a sum of cycles. There exists an algorithm with a time complexity of* $O(k^2 + \alpha_1)$ *to determine if* $A$ *divides* $B$*. This algorithm also describes all quotients* $X = \frac{B}{A}$.

# Example 1: Finding Solutions for $X$

**Equation:** $AX = B$

$$A = 2C_3 + C_9, \quad B = 80C_3 + 40C_9$$

**Solution:**

- By Lemma 3, $X$ is of the form:

$$X = x_1 C_1 + x_3 C_3 + x_9 C_9$$

- System of equations:

$$x_1 + 3x_3 = \frac{80}{2} = 40,$$
$$x_9 = \frac{1}{9}\left(\frac{3 \times 80 + 9 \times 40}{3 \times 2 + 9} - \frac{800}{2}\right) = 0$$

- There are 14 solutions with $x_3$ ranging from 0 to 13, satisfying:

$$x_1 + 3x_3 = 40$$

- $X = x_1 C_1 + x_3 C_3$

# Example 2: Finding Solutions for $Y$

**Equation:** $AY = B$

$$A = 32 + 8C_3 + 4C_9, \quad B = 576 + 704C_3 + 192C_9$$

**Solution:**

- By Lemma 3, $Y$ is of the form:

$$Y = y_1 C_1 + y_3 C_3 + y_9 C_9$$

- System of equations:

$$\begin{aligned} y_1 &= \frac{576}{32} = 18, \\ y_1 + 3y_3 &= \frac{576 + 3 \times 704}{32 + 8 \times 3} = 48, \\ y_9 &= \frac{1}{9}\left(\frac{576 + 3 \times 704 + 9 \times 192}{32 + 8 \times 3 + 4 \times 9} - 48\right) = 0 \end{aligned}$$

- Unique solution: $y_1 = 18$, $y_3 = 10$
- $Y = 18C_1 + 10C_3$

# General Sum of Cycles

**Theorem**

*Let $B$ be a general sum of cycles. Let $A$ be a sum of cycles. There exists an algorithm to determine whether $A$ is a divisor of $B$. In the affirmative case, the algorithm also describes all divisors $X$ such that $A \cdot X = B$.*

# Semiring of Multi Variables

**Definition:** Let $p_1, p_2, \ldots, p_k$ be $k$ distinct prime numbers. Define $\mathbb{N}[p_1, p_2, \ldots, p_k]$ (or $\mathbb{Q}[p_1, p_2, \ldots, p_k]$) as the set of sums of cycles $X$ of the form:

$$X = \sum_{0 \leq i_1, i_2, \ldots, i_k} x_{i_1, i_2, \ldots, i_k} C_{p_1^{i_1}} C_{p_2^{i_2}} \ldots C_{p_k^{i_k}}$$

where $x_{i_1, i_2, \ldots, i_k}$ are coefficients in $\mathbb{N}$ (or $\mathbb{Q}$).

**Operators:** Addition and Multiplication are well-defined over these sets.

**Extension:** The semiring $\mathbb{N}[p_1, p_2, \ldots, p_k, p_{k+1}]$ can be viewed as:

$$\mathbb{N}[p_1, p_2, \ldots, p_k][p_{k+1}]$$

# Algorithm for $m$ Primes

1. Determine the semiring of polynomials:
   - Let $p_1, \dots, p_m$ be the primes in the cycle lengths of $B$.
   - Write $B$ and $A$ in $\mathbb{N}[p_1, p_2, \dots, p_m]$.
   - If $m = 0$ or $m = 1$, apply the algorithm for the single case.
   - Otherwise, continue to Step 2.
2. If $m \geq 2$ and the algorithm for $m - 1$ primes has been solved:
   - Let $k = m - 1$, express $B$ and $A$ in $\mathbb{N}[p_1, p_2, \dots, p_k][p]$, where $p = p_m$.
   - Find $X$ such that $B = A \times X$:

$$B = b_0 C_1 + \sum_i b_{p^{\alpha_i}} C_{p^{\alpha_i}}$$

$$A = A_* + \sum_i a_{p^{\alpha_i}} C_{p^{\alpha_i}}, \quad X = X_* + \sum_i x_{p^{\alpha_i}} C_{p^{\alpha_i}}$$

   with $b_i, a_i, x_i \in \mathbb{N}[p_1, p_2, \dots, p_k]$.

# Find the Suitable Sequence $Q_i$

**Challenge: The $Q_i$ may not be unique.**

- Compute the quotient $Q_k = \frac{n(B)}{n(A)}$ in $\mathbb{N}[p_1, p_2, \ldots, p_k]$, using Algorithm for $k$ primes.

$$Q_k = \frac{n(B)}{n(A)} = \frac{b_1 + p^{\alpha_1} b_{\alpha_1} + \cdots + p^{\alpha_k} b_{\alpha_k}}{a_* + p^{\alpha_1} a_{\alpha_1} + \cdots + p^{\alpha_k} a_{\alpha_k}}.$$

- Let $S_k$ be the set of all posible quotients $Q_k$.
- For each $i = k-1, k-2, \ldots, 2, 1$, find all suitable elements:

$$Q_i = \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}} + \cdots + p^{\alpha_i} b_{p^{\alpha_i}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}} + \cdots + p^{\alpha_i} a_{p^{\alpha_i}}}.$$

- Check the suitable conditions:
  - If there exists $Q_k$ in $S_k$ such that $Q_i \equiv Q_k \mod p^{\alpha_{i+1}}$, and
  - If there exists $Q_{i+1}$ marked by $Q_k$ such that $Q_i \leq Q_{i+1}$.

# Find the Suitable Sequence (continued)

1. Check the suitable conditions:
   - If there exists $Q_k$ in $S_k$ such that $Q_i \equiv Q_k \mod p^{\alpha_{i+1}}$, and
   - If there exists $Q_{i+1}$ marked by $Q_k$ such that $Q_i \leq Q_{i+1}$.
2. If $Q_i$ satisfies these two conditions, then mark $Q_i$ by $Q_k$.
3. Add $Q_i$ to $S_i$.
4. Delete all elements in $S_k$ that are not used to make any $Q_i$ in this step.

**From suitable sequences to solutions.**

- After Step 2.2, we obtain $S_1$, containing all $Q_1$ which have a suitable list $(Q_1, Q_2, Q_3, \ldots, Q_{k-1}, Q_k)$.
- For each suitable list $(Q_1, Q_2, Q_3, \ldots, Q_{k-1}, Q_k)$, we have a solution:

$$\forall i \in \{1, 2, \ldots, k\}, \quad x_{p^{\alpha_i}} = \frac{Q_i - Q_{i-1}}{p^{\alpha_i}},$$

which are in $\mathbb{N}[p_1, p_2, \ldots, p_k]$.

# Example: Find $X$ such that $A \cdot X = B$

Given:

$$B = 72C_8 + 80C_{12} + 48C_{24} + 40C_{36} + 4C_{72},$$
$$A = 4C_8 + 2C_{12} + C_{36}.$$

Consider $p = 3, q = 2$, and $B \in \mathbb{N}[3][2]$.
Express $B$ and $A$ as:

$$B = (80C_3 + 40C_9)C_{2^2} + (72 + 48C_3 + 4C_9)C_{2^3},$$
$$A = (2C_3 + C_9)C_{2^2} + 4C_{2^3}.$$

Form of $X$:

$$X = y_1C_1 + y_2C_2 + y_4C_{2^2} + y_8C_{2^3},$$

where $y_1, y_2, y_4, y_8 \in \mathbb{N}[3]$.

# Example (continued): Using the Algorithm

Using the algorithm:

$$y_1 + 2y_2 + 4y_4 = \frac{80C_3 + 40C_9}{2C_3 + C_9},$$

$$y_8 = \frac{1}{8}\left(\frac{576 + 704C_3 + 192C_9}{32 + 8C_3 + 4C_9} - \frac{80C_3 + 40C_9}{2C_3 + C_9}\right).$$

First compute:

$$\frac{576 + 704C_3 + 192C_9}{32 + 8C_3 + 4C_9} = 18C_1 + 10C_3.$$

Then:

$$y_1 + 2y_2 + 4y_4 = \frac{80C_3 + 40C_9}{2C_3 + C_9} = r_1C_1 + r_3C_3,$$

with $r_1 + 3r_3 = 40$.

# Example (continued): Solutions

Compute $y_8$:

$$y_8 = \frac{1}{8}(18C_1 + 10C_3 - (r_1 + r_3C_3)) = \frac{18 - r_1}{8} + \frac{10 - r_3}{8}C_3,$$

where $r_1 + 3r_3 = 40$.
Since $y_8$ must have non-negative integer coefficients:

$$r_3 = 10, \quad r_1 = 10, \quad y_8 = 1.$$

Number of solutions:

$$y_1+2y_2+4y_4 = 10C_1+10C_3, \quad f(10, 2, 2) = 12 \quad \Rightarrow \quad \text{144 solutions.}$$

Example solutions:

- $y_4 = C_1 + C_3,\ y_2 = C_1,\ y_1 = 4C_1 + 6C_3$
- $y_4 = 2C_1 + C_3,\ y_2 = 2C_3,\ y_1 = 2C_1 + 2C_3$
- $y_4 = 2C_1,\ y_2 = 5C_3,\ y_1 = 2C_1$

# Complexity of Algorithms. Algorithm for Single Prime

- **Step 1:** Takes $O(1)$ time.
- **Step 2:** For each $i$, it takes $O(k)$ time.
- **Step 3:** To describe all solutions, it takes $O(1)$ time.

**Total Complexity:** $O(k^2)$ time for a single prime.

**Number of Solutions:** $s\left(\frac{b_{p^{\alpha_1}}}{a_{p^{\alpha_1}}}, p, \alpha_1\right)$,
where $s(n, p, k)$: number of solutions of:

$$x_1 + px_p + p^2 x_{p^2} + \ldots + p^{k-1} x_{p^{k-1}} + x_{p^k} p^k = n.$$

**Time to list Solutions:** $k \cdot s(n, p, k)$ time.

# Current and Future Tasks:

- Analyze algorithm complexity for the general case.
- Calculate the number of solutions.
- Find divisors of a given sum of cycles.
- Determine when $X$ is a prime.
- Determine when $X$ is irreducible.

**Upper Bound for** $s(n, p, k)$

- A well-studied function initiated by Mahler.
- Focus on finding the upper bound.

Thank you for your attention!

# Theorem and Algorithm

**Theorem 1:** There exists an algorithm with a time complexity of $O(k^2 + \alpha_1)$ to determine if $A$ divides $B$, and to describe all quotients $X = \frac{B}{A}$.

**Algorithm Steps:**

- Step 1: Handle the first condition
- Step 2: Compute all $x_{p^i}$ for $i \geq 1$
- Step 3: Solve the equation

# Algorithm Complexity

**Step 1**: $O(1)$ time.
**Step 2**: For each $i$, takes $O(k)$ time.
**Step 3**: Describing all solutions takes $O(1)$ time.
There are $s\left(\frac{b_{p^{\alpha_1}}}{a_{p^{\alpha_1}}}, p, \alpha_1\right)$ solutions. Listing all solutions takes $k \cdot s(n, p, k)$ time.

# Lemma: Form of $A$ and $X$

**Lemma**

*If $B = A \cdot X$, then:*

$$A_* X_* = b_1 C_1$$

$$(a_* + a_{p^{\alpha_1}} C_{p^{\alpha_1}})(x_* + x_{p^{\alpha_1}} C_{p^{\alpha_1}}) = b_1 + b_{p^{\alpha_1}} C_{p^{\alpha}}$$

$$(a_* + a_{p^{\alpha_1}} C_{p^{\alpha_1}} + a_{p^{\alpha_2}} C_{p^{\alpha_2}})(x_* + x_{p^{\alpha_1}} C_{p^{\alpha_1}} + x_{p^{\alpha_2}} C_{p^{\alpha_2}}) = b_1 + b_{p^{\alpha_1}} C_{p^{\alpha}}$$

$\dots$

# Lemma: Computing $X$

If $B = A \cdot X$ then

$$
\begin{aligned}
&A_* X_* &&= b_1 C_1 \\
&(a_* + a_{p^{\alpha_1}} C_{p^{\alpha_1}})(x_* + x_{p^{\alpha_1}} C_{p^{\alpha_1}}) &&= b_1 + b_{p^{\alpha_1}} C_{p^{\alpha_1}} \\
&(a_* + a_{p^{\alpha_1}} C_{p^{\alpha_1}} + a_{p^{\alpha_2}} C_{p^{\alpha_2}})(x_* + x_{p^{\alpha_1}} C_{p^{\alpha_1}} + x_{p^{\alpha_2}} C_{p^{\alpha_2}}) &&= b_1 + b_{p^{\alpha_1}} C_{p^{\alpha_1}} + b_p \\
&\ldots \\
&(a_* + a_{p^{\alpha_1}} C_{p^{\alpha_1}} + \ldots + a_{p^{\alpha_k}} C_{p^{\alpha_k}})(x_* + x_{p^{\alpha_1}} C_{p^{\alpha_1}} + \ldots + x_{p^{\alpha_k}} C_{p^{\alpha_k}}) &&= b_1 + b_{p^{\alpha_1}} C_{p^{\alpha_1}} + \ldots
\end{aligned}
$$

If $b_1 \neq 0$, then it follows that $a_* = a_1$ and $x_* = x_1$.
In all cases:

$$
\begin{aligned}
&a_* x_* &&= b_1 \\
&(a_* + p^{\alpha_1} a_{p^{\alpha_1}})(x_* + p^{\alpha_1} x_{p^{\alpha_1}}) &&= b_1 + p^{\alpha_1} b_{p^{\alpha_1}} \\
&(a_* + p^{\alpha_1} a_{p^{\alpha_1}} + p^{\alpha_2} a_{p^{\alpha_2}})(x_* + p^{\alpha_1} x_{p^{\alpha_1}} + p^{\alpha_2} x_{p^{\alpha_2}}) &&= b_1 + p^{\alpha_1} b_{p^{\alpha_1}} + p^{\alpha_2} b_{p^{\alpha_2}} \\
&\ldots \\
&(a_* + p^{\alpha_1} a_{p^{\alpha_1}} + \ldots + p^{\alpha_k} a_{p^{\alpha_k}})(x_* + p^{\alpha_1} x_{p^{\alpha_1}} + \ldots + p^{\alpha_k} x_{p^{\alpha_k}}) &&= b_1 + p^{\alpha_1} b_{p^{\alpha_1}} + \ldots + p^{\alpha}
\end{aligned}
$$

# Computing $X$

Or equivalently,

$$
\begin{aligned}
a_* x_* &= b_1 \\
x_{p^{\alpha_1}} &= \frac{1}{p^{\alpha_1}} \left( \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}}} - x_* \right) \\
x_{p^{\alpha_2}} &= \frac{1}{p^{\alpha_2}} \left( \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}} + p^{\alpha_2} b_{p^{\alpha_2}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}} + p^{\alpha_2} a_{p^{\alpha_2}}} - \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}}} \right) \\
&\dots \\
x_{p^{\alpha_i}} &= \frac{1}{p^{\alpha_i}} \left( \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}} + \ldots + p^{\alpha_i} b_{p^{\alpha_i}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}} + \ldots + p^{\alpha_i} a_{p^{\alpha_i}}} - \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}} + \ldots + p^{\alpha_{i-1}} b_{p^{\alpha_{i-1}}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}} + \ldots + p^{\alpha_{i-1}} a_{p^{\alpha_{i-1}}}} \right) \\
&\dots \\
x_{p^{\alpha_k}} &= \frac{1}{p^{\alpha_k}} \left( \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}} + \ldots + p^{\alpha_k} b_{p^{\alpha_k}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}} + \ldots + p^{\alpha_k} a_{p^{\alpha_k}}} - \frac{b_1 + p^{\alpha_1} b_{p^{\alpha_1}} + \ldots + p^{\alpha_{k-1}} b_{p^{\alpha_{k-1}}}}{a_* + p^{\alpha_1} a_{p^{\alpha_1}} + \ldots + p^{\alpha_{k-1}} a_{p^{\alpha_{k-1}}}} \right).
\end{aligned}
$$